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On compactness of Orlicz-Sobolev mappings

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Abstract - We establish compactness of continuous mappings of the Orlicz-Sobolev classes $W^{1,\varphi}_{\rm loc}$ with the Calderon type condition on φ and one fixed point.

Key words and phrases : continuous mappings, locally uniform convergence, compactness, Orlicz-Sobolev classes, Calderon type conditions.

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1. Introduction

The present paper is a natural continuation of our last work [20] on the Orlicz–Sobolev classes. Note that these classes are intensively studied in various aspects at present, see e.g. [1], [7], [8], [10], [15], [16], [18], [19], [21], [22], [33] and [34]. Recall that the problem of equicontinuity of mappings in the classes $W^{1,p}$ for $p > n$ was investigated in the well–known paper [5], cf. also [16]. However, the condition $p > n$ is too restrictive as it was cleared already in the plane case, see e.g. [4], [9], [14] and [23], although this condition was natural for quasiconformal mappings, see e.g. [3] and [12].

Recall some definitions related to the Sobolev spaces $W^{1,p}$, $p \in [1,\infty)$. Given an open set U in $\mathbb{R}^n$, $n \geq 2$, $C_0^\infty(U)$ denotes the collection of all functions $\varphi : U \to \mathbb{R}$ with compact support having continuous partial derivatives of any order. Now, let u and $v : U \to \mathbb{R}$ be locally integrable functions. The function v is called the *distributional derivative* u_{x_i} of u in the variable x_i, $i = 1, 2, \ldots, n$, $x = (x_1, x_2, \ldots, x_n)$, if

$$\int\limits_U u\,\varphi_{x_i}\,dm(x) = -\int\limits_U v\,\varphi\;dm(x) \qquad \forall\;\varphi \in C_0^\infty(U)\;. \tag{1.1}$$

Here $dm(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$. The *Sobolev classes* $W^{1,p}(U)$ consist of all functions $u : U \to \mathbb{R}$ in $L^p(U)$ with all distributional derivatives of the first order in $L^p(U)$. A function $u : U \to \mathbb{R}$ belongs to $W^{1,p}_{\rm loc}(U)$ if $u \in W^{1,p}(U_*)$ for every open set U_* with a compact closure in U.

We use the abbreviation $W^{1,p}_{\rm loc}$ if U is either defined by the context or not essential. The similar notion is introduced for vector-functions $f : U \to \mathbb{R}^m$ in the component-wise sense.

The concept of the distributional derivative was introduced by Sobolev, see [32]. It is known that a continuous function f belongs to $W^{1,p}_{\rm loc}$ if and only if $f \in ACL^p$, i.e., if f is locally absolutely continuous on a.e. straight line which is parallel to a coordinate axis and if all the first partial derivatives of f are locally integrable with the power p, see e.g. 1.1.3 in [24].

In what follows, D is a domain in a finite-dimensional Euclidean space. Following Orlicz, see [26], given a convex increasing function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(0) = 0$, denote by L^φ the space of all functions $f : D \to \mathbb{R}$ such that

$$\int\limits_D \varphi\left(\frac{|f(x)|}{\lambda}\right)\, dm(x) < \infty \tag{1.2}$$

for some $\lambda > 0$. L^φ is called the *Orlicz space*. If $\varphi(t) = t^p$, then we write also L^p. In other words, L^φ is the cone over the class of all functions $g : D \to \mathbb{R}$ such that

$$\int\limits_D \varphi\left(|g(x)|\right)\, dm(x) < \infty \tag{1.3}$$

which is also called the *Orlicz class*, see [2].

The *Orlicz-Sobolev class* $W^{1,\varphi}_{\rm loc}(D)$ is the class of locally integrable functions f given in D with the first distributional derivatives whose gradient ∇f has a modulus $|\nabla f|$ that belongs locally in D to the Orlicz class. Note that by definition $W^{1,\varphi}_{\rm loc} \subseteq W^{1,1}_{\rm loc}$. Later on, we also write $f \in W^{1,\varphi}_{\rm loc}$ for a locally integrable vector-function $f = (f_1, \ldots, f_m)$ of n real variables $x_1, \ldots, x_n$ if $f_i \in W^{1,1}_{\rm loc}$ and

$$\int\limits_D \varphi\left(|\nabla f(x)|\right)\, dm(x) < \infty \tag{1.4}$$

where $|\nabla f(x)| = \sqrt{\sum\limits_{i,j}\left(\frac{\partial f_i}{\partial x_j}\right)^2}$. In this paper we use the notation $W^{1,\varphi}_{\rm loc}$ for functions φ without the normalization $\varphi(0) = 0$ as in the usual Orlicz classes.

2. On one Calderon result

First of all, let us formulate and analyze the fundamental Calderon result in [7], p. 208.

Proposition 2.1. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function with $\varphi(0) = 0$ and the condition*

$$A \, : \, = \int\limits_0^\infty \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} \, dt \; < \; \infty \tag{2.1}$$

for a natural number $k \geq 2$ and let $f : D \to \mathbb{R}$ be a continuous function given in a domain $D \subset \mathbb{R}^k$ of the class $W^{1,\varphi}(D)$. Then

$$\operatorname{diam} f(C) \leq \alpha_k \, A^{\frac{k-1}{k}} \left[\, \int\limits_C \varphi\left(|\nabla f|\right) \, dm(x)\right]^{\frac{1}{k}} \tag{2.2}$$

for every cube $C \subset D$ whose edges are oriented along coordinate axes where α_k is a constant depending only on k.

Remark 2.1. Here it is not essential that the fuction φ is (strictly !) increasing. Indeed, let φ is only nondecreasing. Going over, in case of need, to the new function

$$\tilde{\varphi}_\varepsilon(t) := \varphi(t) + \sum_i \varphi_i^{(\varepsilon)}(t)$$

where

$$\varphi_i^{(\varepsilon)}(t) := \varepsilon \, \frac{2^{-i}}{(b_i - a_i)} \int\limits_0^t \chi_i(t) \, dt$$

and χ_i is a numbering of the characteristic functions of the intervals of constancy (a_i, b_i) of the function φ, we see that $\varphi(t) \leqslant \tilde{\varphi}_\varepsilon(t) \leqslant \varphi(t) + \varepsilon$ and, thus, the condition (1.4) on C and the condition (2.1) hold for the (strictly!) increasing function $\tilde{\varphi}_\varepsilon$. Letting $\varepsilon \to 0$, we obtain the estimate (2.2) with the initial function φ, see e.g. Theorem I.12.1 in [30].

The function $(t/\varphi(t))^{1/(k-1)}$ can have a nonintegrable singularity at zero. However, it is clear that the behavior of the function φ about zero is not essential for the estimate (2.2). Indeed, we may apply the estimate (2.2) with the replacements $A \mapsto A_*$ and $\varphi \mapsto \varphi_*$ where

$$A_* := \left[\frac{1}{\varphi(t_*)}\right]^{\frac{1}{k-1}} + \int\limits_{t_*}^\infty \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{k-1}} dt < \infty \tag{2.3}$$

and $\varphi_*(0) = 0$, $\varphi_*(t) \equiv \varphi(t_*)$ for $t \in (0, t_*)$ and $\varphi_*(t) = \varphi(t)$ for $t \geqslant t_*$ if $\varphi(t_*) > 0$. Hence, in particular, the normalization $\varphi(0) = 0$ in Proposition 2.1 evidently has no valuation, too.

3. The main lemma

Recall that a nondecreasing convex function $\varphi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ is called *strictly convex*, see e.g. [28], if

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} \;=\; \infty \,. \tag{3.1}$$

Here the continuity of functions $\varphi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ will be understood in the sense of the topology of the extended positive real axis $\overline{\mathbb{R}^+}$. Set

$$t_0 \;=\; \sup_{\varphi(t)=0} t\,, \qquad t_0 \;=\; 0 \quad \text{if} \quad \varphi(t)>0 \quad \forall t\in\overline{\mathbb{R}^+} \tag{3.2}$$

and

$$T_0 \;=\; \inf_{\varphi(t)=\infty} t\,, \qquad T_0 \;=\; \infty \quad \text{if} \quad \varphi(t)<\infty \quad \forall t\in\overline{\mathbb{R}^+}\,. \tag{3.3}$$

Remark 3.1. Note that a nonconstant continuous nondecreasing convex function $\varphi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ satisfying the condition of Calderon type

$$\int\limits_{t_*}^{\infty} \left(\frac{t}{\varphi(t)}\right)^{\alpha} dt \;<\; \infty \tag{3.4}$$

for some $\alpha > 0$ and $t_* \in (t_0, \infty)$ is strictly convex. Indeed, the slope $\varphi(t)/t$ is a nondecreasing function if φ is convex, see e.g. Proposition I.4.5 in [6]. Hence the condition (3.4) for $\alpha > 0$ implies (3.1).

The proof of the main result, Theorem 4.1 further, will be based on the following lemma.

Lemma 3.1. *Let $\varphi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ be a nonconstant continuous nondecreasing convex function with the condition* (3.4) *for some $\alpha > 0$ and let $\tilde{\alpha} \in (\alpha, \infty)$. Then φ admits the decomposition $\varphi = \psi \circ \tilde{\varphi}$ where ψ and $\tilde{\varphi} : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ are strictly convex and, moreover, $\tilde{\varphi} \le \varphi$ and $\tilde{\varphi}$ satisfies* (3.4) *with the new $\tilde{\alpha}$.*

Proof. Note that the convex function φ is locally Lipschitz on the interval $(0, T_0)$, where T_0 is defined by (3.3), $T_0 > t_0$ by continuity and variability of the function φ. Consequently, φ is locally absolutely continuous and, furthermore, differentiable except a countable collection of points in the given nondegenerate interval and φ' is nondecreasing, see e.g. Corollaries 1-2 and Proposition 8 of Section I.4 in [6]. Thus, denoting by $\varphi'_+(t)$ the function which coincides with $\varphi'(t)$ at the points of differentiability of φ and

$\varphi'_+(t) = \lim\limits_{\tau\to t+0} \varphi'(\tau)$ at the rest points in the interval $[0, T_0)$ and, finally, setting $\varphi'_+(t) = \infty$ for all $t \in [T_0, \infty]$, we have that

$$\varphi(t) \ = \ \varphi(0) \ + \int\limits_0^t \varphi'_+(\tau)\ d\tau \qquad \forall\ t \in \overline{\mathbb{R}^+}\ . \tag{3.5}$$

By monotonicity of the function φ'_+, calculating its averages over the segments $[0, t]$ and $[t/2, t]$, correspondingly, we obtain from (3.5) the two-sided estimate

$$\frac{1}{2}\ \varphi'_+\,(t/2) \ \le \ \frac{\varphi(t) \ - \ \varphi(0)}{t} \ \le \ \varphi'_+(t) \qquad \forall\ t \in \overline{\mathbb{R}^+}\ . \tag{3.6}$$

The inequalities (3.6) show that the condition (3.4) is equivalent to the following

$$I \ : \ = \int\limits_{t_*}^{\infty} \frac{dt}{[\varphi'_+(t)]^{\alpha}} \ < \ \infty\ . \tag{3.7}$$

Again by monotonicity of φ'_+, the condition (3.7) implies that $\varphi'_+(t) \to \infty$ as $t \to \infty$. Thus, $T_* = \sup\limits_{\varphi'_+(t)<1} t$ is finite, $T_* \in [t_0, T_0)$. Set $\lambda = \alpha/\alpha_* \in (0, 1)$.

Consider the functions $\tilde{\varphi}(t) = \int\limits_0^t h(\tau)\, d\tau$ and $\psi(s) = \varphi(0) + \int\limits_0^s H(r)\, dr$ where $h(t) = \varphi'_+(t)$ for $t \in [0, T_*)$ and $h(t) = [\varphi'_+(t)]^{\lambda}$ for $t \in [T_*, \infty]$ and $H(s) = 1$ for $s \in [0, S_*)$, $S_* = \varphi_*(T_*)$, $H(s) = [\varphi'_+(\tilde{\varphi}^{-1}(s))]^{1-\lambda}$ for $s \in [S_*, S_0)$, $S_0 = \varphi_*(T_0)$, and $H(s) = \infty$ for $s \in [S_0, \infty]$.

By the construction, $\tilde{\varphi}(t) \le \varphi(t)$ for all $t \in \overline{\mathbb{R}^+}$, the functions ψ and $\tilde{\varphi}$ as well as $\psi \circ \tilde{\varphi}$ are nondecreasing and convex, see e.g. Proposition 8 of Section I.4 in [6], and

$$\int\limits_{t_*}^{\infty} \frac{dt}{[\tilde{\varphi}'_+(t)]^{\tilde{\alpha}}} \ = \ I \ < \ \infty \tag{3.8}$$

and, thus, $\tilde{\varphi}$ satisfies (3.4) with the new $\tilde{\alpha}$. Moreover, similarly to (3.6)

$$\frac{\psi(s) \ - \ \psi(0)}{s} \ \ge \ \frac{1}{2}\ H\,(s/2) \qquad \forall\ s \in \overline{\mathbb{R}^+} \tag{3.9}$$

where the right hand side converges to ∞ as $s \to \infty$. Thus, ψ is strictly convex.

Finally, simple calculations by the chain rule show that

$$(\psi \circ \tilde{\varphi})'_+(t) \ = \ \psi'_+(\tilde{\varphi}(t)) \cdot \tilde{\varphi}'_+(t) \ = \ \varphi'_+(t)$$

except a countable collection of points in $\overline{\mathbb{R}^+}$, $\psi \circ \tilde{\varphi}(0) = \varphi(0)$ and, consequently, $\psi \circ \tilde{\varphi} \equiv \varphi$ in view of (3.5). □

4. On compactness of Orlicz–Sobolev mappings

Recall definitions related to normal and compact families of mappings in metric spaces. Let (X,d) and (X',d') be metric spaces with distances d and d', respectively. A family $\mathfrak{F}$ of continuous mappings $f : X \to X'$ is said to be *normal* if every sequence of mappings $f_j \in \mathfrak{F}$ has a subsequence f_{j_m} converging uniformly on each compact set $C \subset X$ to a continuous mapping f. If in addition $\mathfrak{F}$ is *closed* with respect to the locally uniform convergence, i.e., $f \in \mathfrak{F}$, then the family is called *compact*.

Normality is closely related to the following notion. A family $\mathfrak{F}$ of mappings $f : X \to X'$ is said to be *equicontinuous at a point* $x_0 \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathfrak{F}$ and $x \in X$ with $d(x,x_0) < \delta$. The family $\mathfrak{F}$ is called *equicontinuous* if $\mathfrak{F}$ is equicontinuous at every point $x_0 \in X$.

Given a domain D in $\mathbb{R}^n$, $n \geq 2$, a nondecreasing function $\varphi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$, $M \in [0,\infty)$ and $x_0 \in D$, denote by $\mathfrak{F}^{\varphi}_M$ the family of all continuous mappings $f : D \to \mathbb{R}^m$, $m \geq 1$, of the class $W^{1,1}_{\rm loc}$ such that $f(x_0) = 0$ and

$$\int\limits_D \varphi\left(|\nabla f|\right)\, dm(x) \leq M\ . \tag{4.1}$$

We also use the notation $\mathfrak{F}^p_M$ for the case of the function $\varphi(t) = t^p$, $p \in [1,\infty)$.

The main result of the present paper is the following.

Theorem 4.1. *Let $\varphi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ be a nonconstant continuous nondecreasing convex function satisfying the condition* (3.4)*. Then $\mathfrak{F}^{\varphi}_M$ is compact with respect to the locally uniform convergence in $\mathbb{R}^n$.*

Proof. First, let us show that mappings in $\mathfrak{F}^{\varphi}_M$ are equicontinuous. Indeed, by Lemma 3.1 φ admits the decomposition $\varphi = \psi \circ \tilde{\varphi}$ where ψ and $\tilde{\varphi} : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ are strictly convex and

$$\int\limits_{t_*}^{\infty} \left(\frac{t}{\tilde{\varphi}(t)}\right)^{\frac{1}{n-1}} dt \ <\ \infty,$$

moreover, $\tilde{\varphi} \leq \varphi$ and hence

$$\int\limits_D \tilde{\varphi}\left(|\nabla f|\right)\, dm(x) \leq M.$$

Given $z_0 \in D$ and $\delta > 0$, denote by $C(z_0,\delta)$ the n–dimensional open cube centered at the point z_0 with edges which are parallel to coordinate axes and whose length is equal to δ. Fix $\varepsilon > 0$. Since the function ψ is strictly convex, the integral of $\tilde{\varphi}(|\nabla f|)$ over $C(z_0,\delta)\subset D$ is arbitrary small at sufficiently small $\delta>0$ for all $f\in\mathfrak{F}_M^{\varphi}$, see e.g. Theorem III.3.1.2 in [28]. Thus, by Proposition 2.1 and Remark 2.1 applied to $\tilde{\varphi}$ we have that $|f(z)-f(z_0)|<\varepsilon$ for all $z\in C(z_0,\delta)$ under some $\delta=\delta(\varepsilon)>0$.

Now, let us show that a family $\mathfrak{F}_M^{\varphi}$ is uniformly bounded on compacta. Indeed, let K be a compactum in D. With no loss of generality we may consider that K is a connected set containing the point x_0 from the definition of $\mathfrak{F}_M^{\varphi}$, see e.g. Lemma 1 in [31]. Let us cover K by the collection of cubes $C(z,\delta_z)$, $z\in K$, where δ_z corresponds to $\varepsilon:=1$ from the first part of the proof. Since K is compact, we can find a finite number of cubes $C_i=C(z_i,\delta_{z_i})$, $i=1,2,\ldots,N$ that cover K. Note that $D_*:=\bigcup\limits_{i=1}^{N}C_i$ is a subdomain of D because K is a connected set. Consequently, each point $z_*\in K$ can be joined with x_0 in D_* by a polygonal curve with ends of its segments at points $x_0,x_1,\ldots,x_k,z_*$ in the given order lying in the cubes with numbers $i_1,\ldots,i_k$, $x_0\in C(z_{i_1},\delta_{z_{i_1}})$, $z_*\in C(z_{i_k},\delta_{z_{i_k}})$ and $x_l\in C_{i_l}\cap C_{i_{l+1}}$, $l=1,\ldots,k-1$, $k\le N-1$. By the triangle inequality we have that

$$|f(z_*)|\le\sum_{l=0}^{k-1}|f(x_l)-f(x_{l+1})|+|f(x_k)-f(z_*)|\le N\,.$$

Since N depends on a compactum K only, it follows that $\mathfrak{F}_M^{\varphi}$ is uniformly bounded on compacta and, consequently, is normal by the Arzela-Ascoli theorem, see e.g. IV.6.7 in [11].

Finally, show that the class $\mathfrak{F}_M^{\varphi}$ is closed. By Remark 3.1 φ is strictly convex and by Theorem III.3.1.2 in [28], for every $\varepsilon>0$, there is $\delta=\delta(\varepsilon)>0$ such that $\int\limits_E|\nabla f|\,dm(x)\le\varepsilon$ for all $f\in\mathfrak{F}_M^{\varphi}$ whenever $m(E)<\delta$. Let $f_j\in\mathfrak{F}_M^{\varphi}$ and $f_j\to f$ locally uniformly as $j\to\infty$. Then by Lemma 2.1 in [29] we have the inclusion $f\in W^{1,1}_{\rm loc}$. Finally, by Theorem 3.3 in Ch. III, § 3.4, of the monograph [27], f satisfies the condition (4.1), i.e., $\mathfrak{F}_M^{\varphi}$ is closed. Thus, the class $\mathfrak{F}_M^{\varphi}$ is compact. □

Corollary 4.1. *The class $\mathfrak{F}_M^{p}$ is compact with respect to the locally uniform convergence for each $p\in(n,\infty)$.*

Proof. It is easy to verify that the function $\varphi(t)=t^p$ satisfies the hypotheses of Theorem 4.1 for an arbitrary number $\alpha\in(1/(p-1),1/(n-1))$. □

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