On mappings in the Orlicz-Sobolev classes

Denis Kovtonyuk, Vladimir Ryazanov, Ruslan Salimov and
Evgeny Sevost'yanov

Communicated by Cabiria Andreian-Cazacu

Abstract - Under a condition of the Calderon type on φ, we show that
continuous mappings f in $W^{1,\varphi}_{\text{loc}}$, in particular, $f \in W^{1,p}_{\text{loc}}$ for $p > n - 1$
have the (N)-property by Lusin on a.e. hyperplane. It is proved on this
basis that under the given condition on φ the homeomorphisms f with finite
distortion in $W^{1,\varphi}_{\text{loc}}$ and, in particular, $f \in W^{1,p}_{\text{loc}}$ for $p > n - 1$ are the so-
called lower Q-homeomorphisms where $Q(x)$ is equal to its outer dilatation
$K_f(x)$. This makes possible to apply our theory of the boundary behavior
of the lower Q-homeomorphisms to homeomorphisms with finite distortion
in the Orlicz-Sobolev classes.

Key words and phrases : the Lusin property, Sobolev classes, Orlicz-
Sobolev classes, mappings of finite distortion, lower Q-homeomorphisms,
boundary behavior.

Mathematics Subject Classification (2010) : primary 30C65; sec-
ondary 30C75.

1. Introduction

In what follows, D is a domain in a finite-dimensional Euclidean space.
Following Orlicz, see [26], given a convex increasing function $\varphi : [0, \infty)$
$\to [0, \infty)$, $\varphi(0) = 0$, denote by L^φ the space of all functions $f : D \to \mathbb{R}$ such
that

$$\int_D \varphi \left(\frac{|f(x)|}{\lambda} \right) dm(x) < \infty \tag{1.1}$$

for some $\lambda > 0$ where $dm(x)$ corresponds to the Lebesgue measure in D.
L^φ is called the *Orlicz space*. If $\varphi(t) = t^p$, then we write also L^p. In other
words, L_φ is the cone over the class of all functions $g : D \to \mathbb{R}$ such that

$$\int_D \varphi \left(|g(x)| \right) dm(x) < \infty \tag{1.2}$$

which is also called the *Orlicz class*, see [3].

The *Orlicz-Sobolev class* $W^{1,\varphi}_{\text{loc}}(D)$ is the class of locally integrable func-
tions f given in D with the first distributional derivatives whose gradient ∇f

belongs locally in D to the Orlicz class. Note that by definition $W^{1,\varphi}_{\text{loc}} \subseteq W^{1,1}_{\text{loc}}$. As usual, we write $f \in W^{1,p}_{\text{loc}}$ if $\varphi(t) = t^p$, $p \geqslant 1$. It is known that a continuous function f belongs to $W^{1,p}_{\text{loc}}$ if and only if $f \in ACL^p$, i.e., if f is locally absolutely continuous on a.e. straight line which is parallel to a coordinate axis, and if the first partial derivatives of f are locally integrable with the power p, see, e.g., 1.1.3 in [24]. The concept of the distributional derivative was introduced by Sobolev [32] in $\mathbb{R}^n$, $n \geqslant 2$, and it is developed under wider settings at present, see, e.g., [28].

Later on, we also write $f \in W^{1,\varphi}_{\text{loc}}$ for a locally integrable vector-function $f = (f_1, \ldots, f_m)$ of n real variables $x_1, \ldots, x_n$ if $f_i \in W^{1,1}_{\text{loc}}$ and

$$\int_D \varphi\left(|\nabla f(x)|\right) \, dm(x) < \infty \tag{1.3}$$

where $|\nabla f(x)| = \sqrt{\sum_{i,j} \left(\frac{\partial f_i}{\partial x_j}\right)^2}$. Note that in this paper we use the notation $W^{1,\varphi}_{\text{loc}}$ for more general functions φ than in the classical Orlicz classes giving up the condition on convexity of φ. Note also that the Orlicz–Sobolev classes are intensively studied in various aspects at present.

Recall that a homeomorphism f between domains D and D' in $\mathbb{R}^n$, $n \geqslant 2$, is called of *finite distortion* if $f \in W^{1,1}_{\text{loc}}$ and

$$\|f'(x)\|^n \leqslant K(x) \cdot J_f(x) \tag{1.4}$$

with a.e. finite function K where $\|f'(x)\|$ denotes the matrix norm of the Jacobian matrix f' of f at $x \in D$, $\|f'(x)\| = \sup_{h \in \mathbb{R}^n, |h|=1} |f'(x) \cdot h|$, and $J_f(x) = \det f'(x)$ is its Jacobian. Later on, we use the notation $K_f(x)$ for the minimal function $K(x) \geqslant 1$ in (1.4), i.e., we set $K_f(x) = \|f'(x)\|^n / J_f(x)$ if $J_f(x) \neq 0$, $K_f(x) = 1$ if $f'(x) = 0$ and $K_f(x) = \infty$ at the rest points.

First this notion was introduced on the plane for $f \in W^{1,2}_{\text{loc}}$ in the work [16]. Later on, this condition was changed by $f \in W^{1,1}_{\text{loc}}$ but with the additional condition $J_f \in L^1_{\text{loc}}$ in the monograph [15]. The theory of the mappings with finite distortion had many successors, see, e.g., a number of references in the monograph [23]. They had as predecessors the mappings with bounded distortion, see [27] and [34], in other words, the quasiregular mappings, see, e.g., [4], [5], [13], [21], [29] and [35].

Note that the above additional condition $J_f \in L^1_{\text{loc}}$ in the definition of the mappings with finite distortion can be omitted for homeomorphisms. Indeed, for each homeomorphism f between domains D and D' in $\mathbb{R}^n$ with

the first partial derivatives a.e. in D, there is a set E of the Lebesgue measure zero such that f satisfies (N)-property by Lusin on $D \setminus E$ and

$$\int_A J_f(x)\,dm(x) = |f(A)| \tag{1.5}$$

for every Borel set $A \subset D \setminus E$, see, e.g., 3.1.4, 3.1.8 and 3.2.5 in [8]. On this base, it is easy by the Hölder inequality to verify, in particular, that if $f \in W^{1,1}_{\mathrm{loc}}$ is a homeomorphism and $K_f \in L^q_{\mathrm{loc}}$ for some $q > n-1$, then also $f \in W^{1,p}_{\mathrm{loc}}$ for some $p > n-1$, that we use further to obtain corollaries.

In this paper $H^k(A)$, $k \geqslant 0$, $\dim_H A$ denote the *k-dimensional Hausdorff measure* and the *Hausdorff dimension*, correspondingly, of a set A in $\mathbb{R}^n$, $n \geqslant 1$. It was shown in [11] that a set A with $\dim_H A = p$ can be transformed into a set $B = f(A)$ with $\dim_H B = q$ for each pair of numbers p and $q \in (0, n)$ under a quasiconformal mapping f of $\mathbb{R}^n$ onto itself, cf. also [1] and [2].

2. Preliminaries

First of all, the following fine property of functions f in the Sobolev classes $W^{1,p}_{\mathrm{loc}}$ was proved in the monograph [12], Theorem 5.5, and can be extended to the Orlicz-Sobolev classes. The statement follows directly from the Fubini theorem and the known characterization of functions in Sobolev's class $W^{1,1}_{\mathrm{loc}}$ in terms of ACL (absolute continuity on lines), see, e.g., Section 1.1.3 in [24].

Proposition 2.1. *Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$, $m = 1, 2, \ldots$, be a mapping in the Orlicz-Sobolev class $W^{1,\varphi}_{\mathrm{loc}}(U)$ with an increasing function $\varphi : [0, \infty) \to [0, \infty)$. Then, for a.e. hyperplane $\mathcal{P}$ which is parallel to a coordinate hyperplane $\mathcal{P}_0$, the restriction of the function f on the set $\mathcal{P} \cap U$ is a function in the class $W^{1,\varphi}_{\mathrm{loc}}(\mathcal{P} \cap U)$.*

Recall also the little-known Fadell theorem in [7] that makes it possible to extend the well-known theorems of Gehring-Lehto-Menchoff in the plane and Väisälä in $\mathbb{R}^n$, $n \geqslant 3$, see, e.g., [9], [25] and [33], on differentiability a.e. of open mappings in Sobolev's classes to the open mappings in Orlicz-Sobolev classes in $\mathbb{R}^n$, $n \geqslant 3$. A mapping $f : \Omega \to \mathbb{R}^n$ is called *open* if the image of every open set in Ω is an open set in $\mathbb{R}^n$.

Proposition 2.2. *Let $f : \Omega \to \mathbb{R}^n$ be a continuous open mapping on an open set Ω in $\mathbb{R}^n$, $n \geqslant 3$. If f has a differential a.e. on Ω with respect to $n-1$ variables, then f has a total differential a.e. on Ω with respect to all n variables.*

Now, the Calderon result in [6], p. 208, can be formulated in the following way.

Proposition 2.3. *Let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing function with the condition*

$$\int\limits_1^\infty \left[\frac{t}{\varphi(t)} \right]^{\frac{1}{k-1}} dt \; < \; \infty \tag{2.1}$$

for a natural number $k \geqslant 2$ and let $f : D \to \mathbb{R}$ be a continuous function given in a domain $D \subset \mathbb{R}^k$ of the class $W^{1,\varphi}(D)$. Then

$$\operatorname{diam} f(C) \leqslant \alpha_k \, A^{\frac{k-1}{k}} \left[\int\limits_C \varphi_* \left(|\nabla f| \right) \, dm(x) \right]^{\frac{1}{k}} \tag{2.2}$$

for every cube $C \subset D$ whose edges are oriented along coordinate axes where α_k is a constant depending only on k,

$$A \; := \; \left[\frac{1}{\varphi(1)} \right]^{\frac{1}{k-1}} \; + \; \int\limits_1^\infty \left[\frac{t}{\varphi(t)} \right]^{\frac{1}{k-1}} dt \; < \; \infty, \tag{2.3}$$

$\varphi_*(0) = 0$, $\varphi_*(t) \equiv \varphi(1)$ *for* $t \in (0,1)$, *and* $\varphi_*(t) = \varphi(t)$ *for* $t \geqslant 1$.

The following statement is also due to Calderon [6].

Lemma 2.1. *Let Ω be an open set in $\mathbb{R}^k$, $k \geqslant 2$, and let $f : \Omega \to \mathbb{R}^m$, $m \geqslant 1$, be a continuous mapping in the class $W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ where $\varphi : [0,\infty) \to [0,\infty)$ is increasing with the condition (2.1). Then f has a total differential a.e. in Ω.*

Combining Lemma 2.1 and Proposition 2.1, we obtain the next statement.

Corollary 2.1. *Let Ω be an open set in $\mathbb{R}^n$, $n \geqslant 3$, and let $f : \Omega \to \mathbb{R}^m$, $m \geqslant 1$, be a continuous mapping in the class $W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ where $\varphi : [0,\infty) \to [0,\infty)$ is increasing and*

$$\int\limits_1^\infty \left[\frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} dt < \infty. \tag{2.4}$$

Then $f : \Omega \to \mathbb{R}^m$ has a total differential a.e. on a.e. hyperplane which is parallel to a coordinate hyperplane.

Combing Corollary 2.1 and Proposition 2.2, we have the following conclusion.

Theorem 2.1. *Let Ω be an open set in $\mathbb{R}^n$, $n \geqslant 3$, and let $f : \Omega \to \mathbb{R}^n$ be a continuous open mapping in the class $W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ where $\varphi : [0, \infty) \to [0, \infty)$ is increasing with the condition (2.4). Then f has a total differential a.e. in Ω.*

Corollary 2.2. *If $f : \Omega \to \mathbb{R}^n$ is a homeomorphism in $W^{1,1}_{\mathrm{loc}}$ with $K_f \in L^p_{\mathrm{loc}}$ for $p > n - 1$, then f is differentiable a.e.*

3. The Lusin and Sard properties on surfaces

Theorem 3.1. *Let Ω be an open set in $\mathbb{R}^k$, $k \geqslant 2$, and let $f : \Omega \to \mathbb{R}^m$, $m \geqslant 1$, be a continuous mapping in the class $W^{1,\varphi}(\Omega)$ where $\varphi : [0, \infty) \to [0, \infty)$ is increasing with the condition (2.1). Then*

$$H^k(f(E)) \leqslant \gamma_{k,m}\, A^{k-1} \int_E \varphi_* \left(|\nabla f| \right)\, dm(x) \tag{3.1}$$

for every measurable set $E \subset \Omega$ and $\gamma_{k,m} = (m\alpha_k)^k$ where α_k and A are constants from (2.2), $\varphi_*(0) = 0$, $\varphi_*(t) \equiv \varphi(1)$ for $t \in (0, 1)$ and $\varphi_*(t) = \varphi(t)$ for $t \geqslant 1$.

Thus, we come to the following conclusion on the Lusin property of mappings in the Orlicz-Sobolev classes.

Corollary 3.1. *Under the hypotheses of Theorem 3.1 the mapping f has the (N)-property of Lusin; furthermore, f is absolutely continuous with respect to the k-dimensional Hausdorff measure.*

We also obtain the following consequence of Theorem 3.1 of the Sard type for mappings in the Orlicz-Sobolev classes, see in addition Theorem VII.3 in [14].

Corollary 3.2. *Under the hypotheses of Theorem 3.1, we have that $H^k(f(E)) = 0$ whenever $|\nabla f| = 0$ on a measurable set $E \subset \Omega$ and hence $\dim_H f(E) \leqslant k$ and also $\dim f(E) \leqslant k - 1$.*

The proof of Theorem 3.1 is based on the following lemma.

Lemma 3.1. *Let Ω be a domain in $\mathbb{R}^k$, $k \geqslant 2$, and let $f : \Omega \to \mathbb{R}^m$, $m \geqslant 1$, be a continuous mapping in the class $W^{1,\varphi}(G)$ where $\varphi : [0, \infty) \to [0, \infty)$ is*

increasing with the condition (2.1). *Then*

$$\operatorname{diam} f(C) \leqslant m\alpha_k \, A_*^{\frac{k-1}{k}} \left[\int\limits_C \varphi_* \left(|\nabla f| \right) \, dm(x) \right]^{\frac{1}{k}} \tag{3.2}$$

for every cube $C \subset \Omega$ whose edges are oriented along coordinate axes.

Proof of Lemma 3.1. Let us prove (3.2) by induction in $m = 1, 2, \ldots$. Indeed, (3.2) holds by Proposition 2.3 for $m = 1$ and α_k from (2.2). Let us assume that (3.2) is valid for some $m = l$ and prove it for $m = l+1$. Consider an arbitrary vector $\vec{V} = (v_1, v_2, \ldots, v_l, v_{l+1})$ in $\mathbb{R}^{l+1}$ and the vectors $\vec{V_1} = (v_1, v_2, \ldots, v_l, 0)$ and $\vec{V_2} = (0, \ldots, 0, v_{l+1})$. Then $|\vec{V}| = |\vec{V_1} + \vec{V_2}| \leqslant |\vec{V_1}| + |\vec{V_2}|$. Thus, denoting by $\operatorname{Pr}_1 \vec{V} = \vec{V_1}$ and $\operatorname{Pr}_2 \vec{V} = \vec{V_2}$ the projections of vectors from $\mathbb{R}^{l+1}$ onto the coordinate hyperplane $y_{l+1} = 0$ and on the $(l+1)$th axis in $\mathbb{R}^{l+1}$, correspondingly, we obtain that $\operatorname{diam} f(C) \leqslant \operatorname{diam} \operatorname{Pr}_1 f(C) + \operatorname{diam} \operatorname{Pr}_2 f(C)$ and, applying (3.2) under $m = l$ and $m = 1$, we come by monotonicity of φ to the inequality (3.2) under $m = l + 1$. The proof is complete. □

Proof of Theorem 3.1. In view of countable additivity of integral and measure we may assume with no loss of generality that E is bounded and $\overline{E} \subset G$, i.e., $\overline{E}$ is a compactum in G. For each $\varepsilon > 0$ there is an open set $\Omega \subset G$ such that $E \subset \Omega$ and $|\Omega \setminus E| < \varepsilon$, see, e.g., Theorem III (6.6) in [31]. By the above remark we may assume that $\overline{\Omega}$ is a compactum and, thus, the mapping f is uniformly continuous in Ω. Hence Ω can be covered by a countable collection of closed oriented cubes C_i whose interiorities are mutually disjoint and such that $\operatorname{diam} f(C_i) < \delta$ for any prescribed $\delta > 0$ and $\left| \bigcup\limits_{i=1}^{\infty} \partial C_i \right| = 0$. Thus, by Lemma 3.1 we have that

$$H_\delta^k(f(E)) \leqslant H_\delta^k(f(\Omega)) \leqslant \sum_{i=1}^{\infty} \left[\operatorname{diam} f(C_i) \right]^k \leqslant$$

$$\leqslant \gamma_{k,m} A_*^{k-1} \int\limits_{\Omega} \varphi_* \left(|\nabla f| \right) \, dm(x).$$

Finally, by absolute continuity of the indefinite integral and arbitrariness of ε and $\delta > 0$ we obtain (3.1). □

Combining Proposition 2.1 and Corollary 3.1 we obtain the following statement.

Theorem 3.2. *Let U be an open set in $\mathbb{R}^n$, $n \geqslant 3$, and let $\varphi : [0, \infty) \to [0, \infty)$ is increasing with the condition* (2.4). *Then each continuous mapping $f : U \to \mathbb{R}^m$, $m \geq 1$, in the class $W_{\mathrm{loc}}^{1,\varphi}$ has the (N)-property (furthermore,*

it is locally absolutely continuous) with respect to the $(n-1)$-dimensional Hausdorff measure on a.e. hyperplane $\mathcal{P}$ which is parallel to a fixed coordinate hyperplane $\mathcal{P}_0$. Moreover, $H^{n-1}(f(E)) = 0$ whenever $|\nabla f| = 0$ on $E \subset \mathcal{P}$ for a.e. such $\mathcal{P}$.

Note that, if the condition (2.4) holds for an increasing function φ, then the function $\varphi_* = \varphi(ct)$ for $c > 0$ also satisfies (2.4). Moreover, the Hausdorff measures are quasi-invariant under quasi-isometries. By the Lindelöf property of $\mathbb{R}^n$, $U \setminus \{x_0\}$ can be covered by a countable collection of open segments of spherical rings in $U \setminus \{x_0\}$ centered at x_0 and each such segment can be mapped onto a rectangular oriented segment of $\mathbb{R}^n$ by some quasi-isometry, see, e.g., I.5.XI in [20] for the Lindelöf theorem. Thus, applying piecewise Theorem 3.2, we obtain the following.

Corollary 3.3. *Under* (2.4) *each* $f \in W^{1,\varphi}_{\mathrm{loc}}$ *has the* (N)*-property (furthermore, it is locally absolutely continuous) on a.e. sphere S centered at a prescribed point $x_0 \in \mathbb{R}^n$. Moreover, $H^{n-1}(f(E)) = 0$ whenever $|\nabla f| = 0$ on $E \subseteq S$ for a.e. such sphere S.*

4. Lower Q-homeomorphisms and Orlicz-Sobolev classes

The following lemma is key for our further research, see the technical notion of lower Q-homeomorphisms in the paper [17] or in the monograph [23].

Theorem 4.1. *Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 3$, and let $\varphi : [0,\infty) \to [0,\infty)$ be increasing with the condition* (2.4). *Then each homeomorphism $f : D \to D'$ of finite distortion in the class $W^{1,\varphi}_{\mathrm{loc}}$ is a lower Q-homeomorphism at every point $x_0 \in \overline{D}$ with $Q(x) = K_f(x)$.*

Proof. Let B be the (Borel) set of all points $x \in D$ where f has a total differential $f'(x)$ and $J_f(x) \neq 0$. Then, applying Kirszbraun's theorem and uniqueness of approximate differential, see, e.g., 2.10.43 and 3.1.2 in [8], we see that B is the union of a countable collection of Borel sets B_l, $l = 1, 2, \ldots$, such that $f_l = f|_{B_l}$ are bi-Lipschitz homeomorphisms, see, 3.2.2, 3.1.4 and 3.1.8 in [8]. With no loss of generality, we may assume that the B_l are mutually disjoint. Denote also by B_* the set of all points $x \in D$ where f has the total differential but with $f'(x) = 0$.

By the construction the set $B_0 := D \setminus (B \bigcup B_*)$ has Lebesgue measure zero, see Theorem 2.1. Hence by Theorem 2.4 in [18] or by Theorem 9.1 in [23] the area $\mathcal{A}_S(B_0) = 0$ for a.e. hypersurface S in $\mathbb{R}^n$ and, in particular, for a.e. sphere $S_r := S(x_0, r)$ centered at a prescribed point $x_0 \in \overline{D}$. Thus, by Corollary 3.3 $\mathcal{A}_{S_r^*}(f(B_0)) = 0$ as well as $\mathcal{A}_{S_r^*}(f(B_*)) = 0$ for a.e. S_r where $S_r^* = f(S_r)$.

Let Γ be the family of all intersections of the spheres S_r, $r \in (\varepsilon, \varepsilon_0)$, $\varepsilon_0 < d_0 = \sup\limits_{x \in D} |x - x_0|$, with the domain D. Given $\varrho_* \in \mathrm{a}dm\, f(\Gamma)$, $\varrho_* \equiv 0$ outside of $f(D)$, set $\varrho \equiv 0$ outside of D and on B_0,

$$\varrho(x) \; : \; = \; \varrho_*(f(x)) \| f'(x) \| \qquad \text{for } x \in D \setminus B_0 \, .$$

Arguing piecewise on B_l, $l = 1, 2, \ldots$, we have by 1.7.6 and 3.2.2 in [8] that

$$\int\limits_{S_r} \varrho^{n-1} \, d\mathcal{A} \; \geqslant \; \int\limits_{S_*^r} \varrho_*^{n-1} \, d\mathcal{A} \; \geqslant \; 1$$

for a.e. S_r and, thus, $\varrho \in \mathrm{ext\, adm}\,\Gamma$.

The change of variables on each B_l, $l = 1, 2, \ldots$, see, e.g., Theorem 3.2.5 in [8], and countable additivity of integrals give the estimate

$$\int\limits_{D} \frac{\varrho^n(x)}{K_f(x)} \, dm(x) \; \leqslant \; \int\limits_{f(D)} \varrho_*^n(x) \, dm(x)$$

and the proof is complete. $\square$

Corollary 4.1. *Each homeomorphism f of finite distortion in $\mathbb{R}^n$, $n \geqslant 3$, in the class $W_{\mathrm{loc}}^{1,p}$ for $p > n - 1$ is a lower Q-homeomorphism at every point $x_0 \in \overline{D}$ with $Q(x) = K_f(x)$.*

Corollary 4.2. *In particular, each homeomorphism f in $\mathbb{R}^n$, $n \geqslant 3$, with $K_f \in L_{\mathrm{loc}}^q$ for $q > n - 1$ is a lower Q-homeomorphism at every point $x_0 \in \overline{D}$ with $Q(x) = K_f(x)$.*

5. The boundary behavior

The definitions of strongly accessible and weakly flat boundaries can be found in [17] or in [23]. Note that all known regular domains as convex, smooth, Lipschitz, uniform and QED (quasiextremal distance) by Gehring–Martio have weakly flat and, consequently, strongly accessible boundaries and are locally connected on their boundaries, see, e.g., Lemma 5.1 in [17] or Lemma 3.15 in [23].

In view of Theorem 4.1, we have by Lemma 6.1 and Theorem 10.1 in [17] or Lemma 9.4 and Theorem 9.6 in [23] the following result.

Theorem 5.1. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geqslant 3$, and let $f : D \to D'$ be a homeomorphism of finite distortion in $W_{\mathrm{loc}}^{1,\varphi}$ where $\varphi : [0, \infty) \to [0, \infty)$ is increasing with the condition (2.4). Suppose that the*

domain D is locally connected on ∂D and that the domain D' has a (strongly accessible) weakly flat boundary. If

$$\int\limits_{0}^{\delta(x_0)} \frac{dr}{\|K_f\|_{n-1}(x_0, r)} = \infty \qquad \forall\, x_0 \in \partial D \tag{5.1}$$

for some $\delta(x_0) \in (0, d(x_0))$ where $d(x_0) = \sup\limits_{x \in D} |x - x_0|$ and

$$\|K_f\|_{n-1}(x_0, r) = \left(\int\limits_{D \cap S(x_0, r)} K_f^{n-1}(x)\, d\mathcal{A} \right)^{\frac{1}{n-1}},$$

then f has a (continuous) homeomorphic extension $\overline{f}$ to $\overline{D}$ mapping $\overline{D}$ (into) onto $\overline{D'}$.

In particular, as a consequence of Theorem 5.1 we obtain the following generalization of the well-known Gehring–Martio theorem on a homeomorphic extension to the boundary of quasiconformal mappings between QED domains, see [10], cf. [22].

Corollary 5.1. *Let D and D' be bounded domains with weakly flat boundaries in $\mathbb{R}^n$, $n \geqslant 3$, and let $f : D \to D'$ be a homeomorphism of finite distortion in the class $W_{\mathrm{loc}}^{1,p}$, $p > n-1$, in particular, a homeomorphism in the class $W_{\mathrm{loc}}^{1,1}$ with $K_f \in L_{\mathrm{loc}}^q$, $q > n-1$. If the condition (5.1) holds, then f has a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

The continuous extension to the boundary of the inverse mappings has a simpler criterion. Namely, in view of Theorem 4.1, we have by Theorem 9.1 in [17] or Theorem 9.6 in [23] the next statement.

Theorem 5.2. *Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 3$, D be locally connected on ∂D and $\partial D'$ be weakly flat. If f is a homeomorphism of finite distortion of D onto D' in the class $W_{\mathrm{loc}}^{1,\varphi}$ where $\varphi : [0, \infty) \to [0, \infty)$ is increasing with the condition (2.4) and $K_f \in L^{n-1}(D)$, then f^{-1} has an extension to $\overline{D'}$ by continuity in $\overline{\mathbb{R}^n}$.*

However, as it follows from the example in Proposition 6.3 from [23], any degree of integrability $K_f \in L^q(D)$, $q \in [1, \infty)$, cannot guarantee the extension by continuity to the boundary of the direct mappings.

Finally, in view of Theorem 4.1, by Theorem 5.1 above and Theorem 2.1 in [30] under $p = n-1$, we obtain the following result.

Theorem 5.3. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n \geqslant 3$, D be locally connected on ∂D and D' have (strongly accessible) weakly flat boundary. Suppose $f : D \to D'$ is a homeomorphism of finite distortion in D in the class $W^{1,\varphi}_{\mathrm{loc}}$ with the condition (2.4) such that*

$$\int\limits_D \Phi(K_f^{n-1}(x))\,dm(x) < \infty \tag{5.2}$$

for a convex increasing function $\Phi : [0, \infty] \to [0, \infty]$. If, for some $\delta > \Phi(0)$,

$$\int\limits_\delta^\infty \frac{d\tau}{\tau\,[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty, \tag{5.3}$$

then f has a (continuous) homeomorphic extension $\overline{f}$ to $\overline{D}$ mapping $\overline{D}$ (into) onto $\overline{D'}$.

Remark 5.1. Note that the condition (5.3) is not only sufficient but also necessary for continuous extension to the boundary of f with the integral constraints (5.2), see, e.g., [19].

Note also that by Theorem 2.1 in [30] under $p = n - 1$ the condition (5.3) is equivalent to the following condition

$$\int\limits_\delta^\infty \log \Phi(t)\,\frac{dt}{t^{n'}} = +\infty \tag{5.4}$$

for some $\delta > 0$ where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e., $n' = 2$ for $n = 2$, n' is strictly decreasing in n and $n' = n/(n-1) \to 1$ as $n \to \infty$.

References

[1] Z.M. Balogh, Hausdorff dimension distribution of quasiconformal mappings on the Heisenberg group, *J. Anal. Math.*, **83** (2001), 289-312.

[2] C.J. Bishop, Quasiconformal mappings which increase dimension, *Ann. Acad. Sci. Fenn. Math.*, **24** (1999), 397-407.

[3] Z. Birnbaum and W. Orlicz, Über die Verallgemeinerungen des Begriffes der zueinauder konjugierten Potenzen, *Studia Math.*, **3** (1931), 1-67.

[4] B. Bojarski and T. Iwaniec, Analytical foundations of the theory of quasiconformal mappings in $\mathbb{R}^n$, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **8** (1983), 257-324.

[5] M. Bonk and J. Heinonen, Smooth quasiregular mappings with branching, *Publ. Math. Inst. Hautes Études Sci.*, **100** (2004), 153-170.

[6] A.P. Calderon, On the differentiability of absolutely continuous functions, *Rivista Mat. Univ. Parma*, **2** (1951), 203-213.

[7] A.G. Fadell, A note on a theorem of Gehring and Lehto, *Proc. Amer. Math. Soc.*, **49** (1975), 195-198.

[8] H. FEDERER, *Geometric Measure Theory*, Springer-Verlag, Berlin 1969.

[9] F.W. GEHRING and O. LEHTO, On the total differentiability of functions of a complex variable, *Ann. Acad. Sci. Fenn. Ser. A I*, **272** (1959), 3-8.

[10] F.W. GEHRING and O. MARTIO, Quasiextremal distance domains and extension of quasiconformal mappings, *J. Anal. Math.*, **45** (1985), 181-206.

[11] F.W. GEHRING and J. VÄISÄLÄ, Hausdorff dimension and quasiconformal mappings, *J. London Math. Soc. (2)*, **6** (1973), 504-512.

[12] V.M. GOLDSHTEIN and YU.G RESHETNYAK, *Introduction to the Theory of Functions with Distributional Derivatives and Quasiconformal Mappings*, Nauka, Moscow, 1983; English translation, *Quasiconformal Mappings and Sobolev Spaces*, Kluwer, Dordrecht, 1990.

[13] J. HEINONEN, T. KILPELAINEN and O. MARTIO, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Clarendon Press, Oxford-New York-Tokio, 1993.

[14] W. HUREWICZ and H. WALLMAN, *Dimension Theory*, Princeton Univ. Press, Princeton, N.J., 1948.

[15] T. IWANIEC and G. MARTIN, *Geometric Function Theory and Non-linear Analysis*, Oxford Math. Monogr., Oxford Univ. Press, Oxford, 2001.

[16] T. IWANIEC and V. SVERAK, On mappings with integrable dilatation, *Proc. Amer. Math. Soc.*, **118** (1993), 181-188.

[17] D. KOVTONYUK and V. RYAZANOV, On the theory of lower Q-homeomorphisms, *Ukr. Mat. Visn.* **5** (2008), 159-184; translation in *Ukr. Math. Bull.*, **5** (2008), 157-181.

[18] D. KOVTONYUK and V. RYAZANOV, On the theory of mappings with finite area distortion, *J. Anal. Math.*, **104** (2008), 291-306.

[19] D. KOVTONYUK and V. RYAZANOV, On the boundary behavior of generalized quasi-isometries, *J. Anal. Math.*, **115** (2011), 103-120 .

[20] K. KURATOWSKI, *Topology*, vol. 1. Acad. Press, N.Y., 1968.

[21] O. MARTIO, S. RICKMAN and J. VAISALA, Definitions for quasiregular mappings, *Ann. Acad. Sci. Fenn. Ser. A I*, **448** (1969), 1-40.

[22] O. MARTIO and M. VUORINEN, Whitney cubes, p-capacity and Minkowski content, *Exposition. Math.*, **27** (1987), 17-40.

[23] O. MARTIO, V. RYAZANOV, U. SREBRO and E. YAKUBOV, *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics, Springer, New York, 2009.

[24] V MAZ'YA, *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.

[25] D MENCHOFF, Sur les differencelles totales des fonctions univalentes, *Math. Ann.*, **105** (1931), 75-85.

[26] W ORLICZ, Über eine gewisse Klasse von Räumen vom Typus B, *Bull. Int. Acad. Pol. Ser. A*, **8** (1932), 207-220.

[27] YU.G. RESHETNYAK, *Space Mappings with Bounded Distortion*, Nauka, Novosibirsk, 1982; English translation, Translations of Mathematical Monographs, vol. 73, Amer. Math. Soc., Providence, R.I., 1988.

[28] YU.G. RESHETNYAK, Sobolev classes of functions with values in a metric space, *Sibirsk. Mat. Zh.*, **38** (1997), 657-675; translation in *Siberian Math. J.*, **38** (1997), 567-583.

[29] S. RICKMAN, *Quasiregular Mappings*, Springer, Berlin, 1993.

[30] V. Ryazanov, U. Srebro and E. Yakubov, On integral conditions in the mapping theory, *Ukr. Mat. Visn.* **7** (2010), 73-87; translation in *J. Math. Sci.*, **173** (2011), 397-407.

[31] S. Saks, *Theory of the Integral*, Dover, New York, 1964.

[32] S.L Sobolev, *Applications of Functional Analysis in Mathematical Physics*, Izdat. Gos. Univ., Leningrad, 1950; English translation Amer. Math. Soc., Providence, R.I., 1963.

[33] J. Väisälä, On quasiconformal mappings in space, *Ann. Acad. Sci. Fenn. Ser. A I*, **298** (1961), 1-36 .

[34] S Vodop'yanov, Mappings with bounded distortion and with finite distortion on Carnot groups, *Sibirsk. Mat. Zh.*, **40** (1999), 764-804; translation in *Siberian Math. J.*, **40** (1999), 644-677.

[35] S. Vuorinen, *Conformal Geometry and Quasiregular mappings*, Lecture Notes in Math, vol. 1319, Springer-Verlag, Berlin, 1988.

Denis Kovtonyuk, Vladimir Ryazanov, Ruslan Salimov and Evgeny Sevost'yanov

Institute of Applied Mathematics and Mechanics,

National Academy of Sciences of Ukraine

74 Roze Luxemburg str., 83114 Donetsk, Ukraine

E-mail: `denis_kovtonyuk@bk.ru, vlryazanov1@rambler.ru,`

`salimov@rambler.ru, e_sevostyanov@rambler.ru`