

Topological mappings of integrally bounded p -moduli

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Dedicated to Professor Vladimir I. Ryazanov on the occasion of his 60th birthday.

Abstract - We study homeomorphisms of controlled p -module by certain integrals. In this way, we establish various properties of mappings and show that their features are close to quasiconformal and bilipschitz mappings.

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1. Introduction

A great interest to studying various classes of homeomorphisms and more general mappings is motivated by needs of many fields in Modern Mathematics. Some of basic classes are close to quasiconformal and bilipschitz homeomorphisms. A main characterization of such mappings is obtained by extension of quasiinvariance of the conformal moduli and p -moduli via inequalities containing integrals depending on a given measurable functions and admissible metrics (cf. [3, 8, 9, 13, 18, 22, 23, 24]). Such representation of moduli can be treated as the quasiinvariance of weighted moduli (cf. [1, 2, 5, 25]).

Let $f : G \rightarrow G^*$, $G, G^* \subset \mathbb{R}^n$, be a homeomorphism such that f and f^{-1} are differentiable almost everywhere (a.e.) with nonzero Jacobians in G and G^* , respectively. It was shown in [9], that under more restrictive conditions on f the following bounds for the α -module of k -dimensional surface families

$$\inf_{\varrho \in \text{extadm } \mathcal{S}_k} \int_G \frac{\varrho^\alpha(x)}{H_{O,\alpha}(x, f)} dx \leq \mathcal{M}_\alpha(f(\mathcal{S}_k)) \leq \inf_{\rho \in \text{adm } \mathcal{S}_k} \int_G \rho^\alpha(x) H_{I,\alpha}(x, f) dx \quad (1.1)$$

are fulfilled. Here $H_{I,\alpha}(x, f)$ and $H_{O,\alpha}(x, f)$ stand for the α -inner and α -outer dilatations of f at $x \in G$ (see, e.g., [7]).

In the case, when these dilatations are bounded, i.e. $H_{I,\alpha}(x, f) \leq K$ and $H_{O,\alpha}(x, f) \leq K$ with some absolute constant K in G , one obtains the well-known class of bilipschitz mappings in G (cf. [6, 7, 10]).

In this paper, we consider the homeomorphisms satisfying at least one of the following conditions

$$\mathcal{M}_\alpha(f(\mathcal{S}_k)) \leq \inf_{\rho \in \text{adm } \mathcal{S}_k} \int_G \rho^\alpha(x) Q(x) dx, \quad (1.2)$$

$$\mathcal{M}_\alpha(f(\mathcal{S}_k)) \geq \inf_{\varrho \in \text{extadm } \mathcal{S}_k} \int_G \frac{\varrho^\alpha(x)}{Q(x)} dx, \quad (1.3)$$

with a given measurable function $Q : G \rightarrow [0, \infty]$. For such mappings the problem can be formulated somewhat similarly to the classical problem on the properties of solutions to the Beltrami equation $f_{\bar{z}} = \mu(z)f_z$, for which the properties of f are investigated in their dependence on the features of μ .

The main cases in (1.2)-(1.3) relate to $k = 1$ and $k = n - 1$, i.e. to moduli of curve and of $(n - 1)$ -surface families. We show that inequality (1.2) yields differentiability a.e., the (N) -property, boundedness of the α -inner dilatation. We also provide the necessary and sufficient condition for a homeomorphism to satisfy (1.3). Finally, we establish the relationship between homeomorphisms satisfying (1.2) for $k = 1$ and (1.3) for $k = n - 1$.

2. Dilatations in \mathbb{R}^n

Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear bijection. The numbers

$$H_{I,\alpha}(A) = \frac{|\det A|}{l^\alpha(A)}, \quad H_{O,\alpha}(A) = \frac{\|A\|^\alpha}{|\det A|}, \quad \alpha \geq 1,$$

are called the α -inner and α -outer dilatations of A , respectively. Here

$$l(A) = \min_{|h|=1} |Ah|, \quad \|A\| = \max_{|h|=1} |Ah|,$$

denote the minimal and maximal stretching of A and $\det A$ is the determinant of A .

Let G and G^* be two bounded domains in \mathbb{R}^n , $n \geq 2$, and let a mapping $f : G \rightarrow G^*$ be differentiable at a point $x \in G$. This means there exists a linear mapping $f'(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, called the (strong) derivative of the mapping f at x , such that

$$f(x+h) = f(x) + f'(x)h + \omega(x, h)|h|,$$

where $\omega(x, h) \rightarrow 0$ as $h \rightarrow 0$.

We denote

$$H_{I,\alpha}(x, f) = H_{I,\alpha}(f'(x)), \quad H_{O,\alpha}(x, f) = H_{O,\alpha}(f'(x)).$$

These quantities naturally extend the classical quasiconformal dilatations (inner and outer) by

$$H_I(x, f) = H_{I,n}(x, f), \quad H_O(x, f) = H_{O,n}(x, f).$$

The third dilatation of quasiconformality called *linear*

$$H(x, f) = \frac{\|f'(x)\|}{l(f'(x))}$$

is a direct analog of the classical planar Lavrentiev dilatation. The α -inner and α -outer dilatations provide a class of mappings whose basic properties are close to quasiconformal homeomorphisms. On the other hand, there are some essential differences caused by the fact that the dilatations $H_I(x, f)$ and $H_O(x, f)$ are always greater than or equal to 1, while the α -inner and outer dilatations range between 0 and ∞ .

We consider the homeomorphisms f which are differentiable almost everywhere in G , and fix the real numbers α, β satisfying $1 \leq \alpha < \beta < \infty$. Define

$$HI_{\alpha,\beta}(f) = \int_G H_{I,\alpha}^{\frac{\beta}{\beta-\alpha}}(x, f) dx, \quad HO_{\alpha,\beta}(f) = \int_G H_{O,\beta}^{\frac{\alpha}{\beta-\alpha}}(x, f) dx,$$

and call these quantities the *inner* and *outer mean dilatations* of a mapping $f : G \rightarrow \mathbb{R}^n$ in G .

Define for the fixed real numbers $\alpha, \beta, \gamma, \delta$ such that $1 \leq \alpha < \beta < \infty$, $1 \leq \gamma < \delta < \infty$, the *class of mappings with finite mean dilatations* which consists of homeomorphisms $f : G \rightarrow G^*$ satisfying:

- (i) f and f^{-1} are in $W_{\text{loc}}^{1,1}$,
- (ii) f and f^{-1} are differentiable, with Jacobians $J(x, f) \neq 0$ and $J(y, f^{-1}) \neq 0$ a.e. in G and G^* , respectively,
- (iii) the inner and the outer mean dilatations $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$ are finite.

The mappings with finite mean dilatations were investigated in [7].

The relations between the classical quasiconformal dilatations

$$\begin{aligned} H(x, f) &\leq \min(H_I(x, f), H_O(x, f)) \leq H^{n/2}(x, f) \\ &\leq \max(H_I(x, f), H_O(x, f)) \leq H^{n-1}(x, f). \end{aligned}$$

show that they are finite or infinity simultaneously. However, this needs not be true for the mean dilatations.

The following example shows that the unboundedness of one from these dilatations does not depend on the value of another mean dilatation.

Consider the unit cube

$$G = \{x = (x_1, \dots, x_n) \mid 0 < x_k < 1, k = 1, \dots, n\}$$

and let

$$f(x) = \left(x_1, \dots, x_{n-1}, \frac{x_n^{1-c}}{1-c}\right), \quad 0 < c < 1.$$

An easy computation shows that f belongs to the class of mappings with finite mean dilatations if and only if

$$0 < c < 1 - \alpha/\beta \quad \text{and} \quad 0 < c < 1 - (\gamma - 1)\delta/(\delta - 1)\gamma.$$

When

$$1 - \alpha/\beta \leq c < 1 \quad \text{and} \quad 1 - (\gamma - 1)\delta/(\delta - 1)\gamma \leq c < 1$$

we have $HI_{\alpha,\beta}(f) = \infty$ and $HO_{\gamma,\delta}(f) = \infty$, respectively. Thus, by suitable choice of parameters $c, \alpha, \beta, \gamma, \delta$, one obtains any desired relations between $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$.

3. α -moduli of k -dimensional surfaces and related classes of homeomorphisms

Now we give a geometric (modular) description of quasiconformality in \mathbb{R}^n starting with the definition of k -dimensional Hausdorff measure H^k , $k = 1, \dots, n - 1$ in \mathbb{R}^n . For a given $E \subset \mathbb{R}^n$, put

$$H^k(E) = \sup_{r>0} H_r^k(E),$$

where

$$H_r^k(E) = \Omega_k \inf \sum_i (\delta_i/2)^k.$$

Here the infimum is taken over all countable coverings $\{E_i, i = 1, 2, \dots\}$ of E with diameters δ_i , and Ω_k is the volume of the unit ball in \mathbb{R}^k .

Let \mathcal{S} be a k -dimensional surface, which means that $\mathcal{S} : D_s \rightarrow \mathbb{R}^n$ is a continuous image of the closed domain $D_s \subset \mathbb{R}^k$. We denote by

$$N(\mathcal{S}, y) = \text{card } \mathcal{S}^{-1}(y) = \text{card}\{x \in D_s \mid \mathcal{S}(x) = y\}$$

the multiplicity function of the surface \mathcal{S} at the point $y \in \mathbb{R}^n$. For a given Borel set $B \subseteq \mathbb{R}^n$, the k -dimensional Hausdorff area of B in \mathbb{R}^n associated with the surface \mathcal{S} is determined by

$$\mathcal{H}_{\mathcal{S}}(B) = \mathcal{H}_{\mathcal{S}}^k(B) = \int_B N(\mathcal{S}, y) dH^k y.$$

If $\rho : \mathbb{R}^n \rightarrow [0, \infty]$ is a Borel function, the integral of ρ over \mathcal{S} is defined by

$$\int_{\mathcal{S}} \rho d\sigma_k = \int_{\mathbb{R}^n} \rho(y) N(\mathcal{S}, y) dH^k y.$$

Let \mathcal{S}_k be a family of k -dimensional surfaces \mathcal{S} in \mathbb{R}^n , $1 \leq k \leq n-1$ (curves for $k=1$). The α -module of \mathcal{S}_k is defined as

$$\mathcal{M}_\alpha(\mathcal{S}_k) = \inf \int_{\mathbb{R}^n} \rho^\alpha dx, \quad \alpha \geq k,$$

where the infimum is taken over all Borel measurable functions $\rho \geq 0$ and such that

$$\int_{\mathcal{S}} \rho^k d\sigma_k \geq 1$$

for every $\mathcal{S} \in \mathcal{S}_k$. We call each such ρ an *admissible function* for \mathcal{S}_k ($\rho \in \text{adm } \mathcal{S}_k$). The n -module $\mathcal{M}_n(\mathcal{S}_k)$ will be denoted by $\mathcal{M}(\mathcal{S}_k)$.

Following [13], a metric ρ is said to be *extensively admissible* for \mathcal{S}_k ($\rho \in \text{extadm } \mathcal{S}_k$) with respect to α -module if $\rho \in \text{adm}(\mathcal{S}_k \setminus \tilde{\mathcal{S}}_k)$ such that $\mathcal{M}_\alpha(\tilde{\mathcal{S}}_k) = 0$.

Accordingly, we say that a property P holds for almost every k -dimensional surface, if P holds for all surfaces except a family of zero α -module.

We also remind that a continuous mapping f satisfies (N)-property with respect to k -dimensional Hausdorff area if $\mathcal{H}_S^k(f(B)) = 0$ whenever $\mathcal{H}_S^k(B) = 0$. Similarly, f has (N^{-1})-property if $\mathcal{H}_S^k(B) = 0$ whenever $\mathcal{H}_S^k(f(B)) = 0$.

Now we provide the bounds for the α -module of k -dimensional surfaces (see [9]).

Theorem 3.1. *Let $f : G \rightarrow \mathbb{R}^n$ be a homeomorphism satisfying (i)-(ii) with $H_{I,\alpha}, H_{O,\alpha}^{-1} \in L_{\text{loc}}^1(G)$. Suppose that for some k , $1 \leq k \leq n-1$ ($k \leq \alpha$), and for almost every k -dimensional surface \mathcal{S} and its image \mathcal{S}^* ($\mathcal{S} = f^{-1}(\mathcal{S}^*)$) the restriction $f|_{\mathcal{S}}$ has the (N) and (N^{-1})-properties with respect to k -dimensional Hausdorff area in G and $G^* = f(G)$, respectively. Then the double inequality (1.1) holds for any family \mathcal{S}_k of k -dimensional surfaces in G , and for each $\rho \in \text{adm } \mathcal{S}_k$ and $\varrho \in \text{extadm } \mathcal{S}_k$ with respect to the α -module.*

Further we use the following lemma from [18].

Lemma 3.1. *Let (X, μ) be a measure space with finite measure μ , and let $\varphi : G \rightarrow (0, \infty)$ be a measurable function. Set*

$$I(\varphi, \alpha) = \inf_{\rho} \int_X \varphi \rho^\alpha d\mu, \quad 1 < \alpha < \infty,$$

where the infimum is taken over all Borel nonnegative measurable functions $\rho : X \rightarrow [0, \infty]$ satisfying $\int_X \rho d\mu = 1$. Then

$$I(\varphi, \alpha) = \left(\int_X \varphi^{\frac{1}{1-\alpha}} d\mu \right)^{1-\alpha}$$

and the infimum is attained only for the metric

$$\rho = \left(\int_X \varphi^{\frac{1}{1-\alpha}} d\mu \right)^{-1} \varphi^{\frac{1}{1-\alpha}}.$$

Throughout the paper, we use the following notations. A ring domain $\mathcal{R} \subset \mathbb{R}^n$ is a bounded domain whose complement consists of two components C_0 and C_1 . The sets $F_0 = \partial C_0$ and $F_1 = \partial C_1$ are two boundary components of D . We assume for definiteness that $\infty \in C_1$.

We say that a curve γ joins the boundary components in \mathcal{R} if γ is located in \mathcal{R} , except for its endpoints, one of which lies on F_0 and the second on F_1 . A compact set Σ is said to separate the boundary components of \mathcal{R} if $\Sigma \subset \mathcal{R}$ and if C_0 and C_1 are located in different components of $C\Sigma$. Denote by $\Gamma_{\mathcal{R}}$ the family of all locally rectifiable curves γ which join the boundary components of \mathcal{R} and by $\Sigma_{\mathcal{R}}$ the family of all compact piecewise smooth $(n-1)$ -dimensional surfaces Σ which separate the boundary components of \mathcal{R} .

The following relation

$$\mathcal{M}_p(\Gamma_{\mathcal{R}}) = \frac{1}{\mathcal{M}_\alpha^{p-1}(\Sigma_{\mathcal{R}})}, \quad \alpha = \frac{p(n-1)}{p-1}, \quad 1 < p < \infty, \quad n-1 < \alpha < \infty, \quad (3.1)$$

between the p -moduli of $\Sigma_{\mathcal{R}}$ and $\Gamma_{\mathcal{R}}$ follows from the results of Ziemer [26] and Hesse [12] on the moduli and the extremal lengths. Observe that p -moduli ($p \neq n$) are not conformal invariants even under linear mappings, i.e. such mappings do not preserve the value of p -module.

The p -module of a spherical ring $A(x_0; a, b) = \{x \in \mathbb{R}^n \mid 0 < a < |x - x_0| < b\}$ is equal to

$$\mathcal{M}_p(\Gamma_A) = \omega_{n-1} \left(\frac{n-p}{p-1} \right)^{p-1} \left(a^{\frac{p-n}{p-1}} - b^{\frac{p-n}{p-1}} \right)^{1-p},$$

where ω_{n-1} is the $(n-1)$ -dimensional Lebesgue measure of the unit sphere S^{n-1} in \mathbb{R}^n (see e.g. [6]). Indeed, for $f(x) = \lambda x$, $\lambda > 0$, $\lambda \in \mathbb{R}$, we have $\mathcal{M}_p(f(\Gamma_A)) = \lambda^{n-p} \mathcal{M}_p(\Gamma_A)$.

We also use especially another tool which is important in Potential Theory and Mathematical Analysis.

Following in general [16], a pair $E = (A, C)$, where $A \subset \mathbb{R}^n$ is an open set and $C \subset A$ is a nonempty compact, is called the *condenser*. We say that the condenser E is the *ring condenser*, if $\mathcal{R} = A \setminus C$ is a ring domain. The condenser E is bounded, if A is bounded. We also say that a condenser $E = (A, C)$ lies in a domain G when $A \subset G$. Obviously, for an open and continuous mapping $f : G \rightarrow \mathbb{R}^n$ and for any condenser $E = (A, C) \subset G$, the pair $(f(A), f(C))$ is a condenser in $f(G)$. In this case we shall use the notation $f(E) = (f(A), f(C))$.

Let $E = (A, C)$ be a condenser. Denote by $\mathcal{C}_0(A)$ the set of all continuous functions $u : A \rightarrow \mathbb{R}^1$ with compact support in A . Consider the set $\mathcal{W}_0(E) = \mathcal{W}_0(A, C)$ of all nonnegative functions $u : A \rightarrow \mathbb{R}^1$ such that
 1) $u \in \mathcal{C}_0(A)$, 2) $u(x) \geq 1$ for $x \in C$ and 3) u belongs ACL. Put

$$\text{cap}_p E = \text{cap}_p(A, C) = \inf_{u \in \mathcal{W}_0(E)} \int_A |\nabla u|^p dx, \quad p \geq 1,$$

where, as usual

$$|\nabla u| = \left(\sum_{i=1}^n (\partial_i u)^2 \right)^{1/2}.$$

This quantity is called *p -capacity of condenser E* .

It was proven in [12] that for $p > 1$

$$\text{cap}_p E = \mathcal{M}_p(\Delta(\partial A, \partial C; A \setminus C)), \quad (3.2)$$

where $\Delta(\partial A, \partial C; A \setminus C)$ denotes the set of all continuous curves which join the boundaries ∂A and ∂C in $A \setminus C$. For general properties of p -capacities and their relation to the mapping theory, we refer for instance to [10] and [19]. In particular, for $1 \leq p < n$,

$$\text{cap}_p E \geq n \Omega_n^{\frac{p}{n}} \left(\frac{n-p}{p-1} \right)^{p-1} [mC]^{\frac{n-p}{n}}, \quad (3.3)$$

where Ω_n denotes the volume of the unit ball in \mathbb{R}^n , and mC is the n -dimensional Lebesgue measure of C .

For $n-1 < p \leq n$, there is the following lower estimate

$$(\text{cap}_p E)^{n-1} \geq \gamma \frac{d(C)^p}{(mA)^{1-n+p}}, \quad (3.4)$$

where $d(C)$ denotes the diameter of C , and γ is a positive constant depending only on n and p (see [14]).

4. Q -homeomorphisms and their properties

Let $Q : G \rightarrow [1, \infty]$ be a measurable function. Due to [17], a homeomorphism $f : G \rightarrow \mathbb{R}^n$ is called a Q -homeomorphism if

$$\mathcal{M}(f(\Gamma)) \leq \int_G Q(x) \rho^n(x) dx \quad (4.1)$$

for every family Γ of curves in G and for every admissible function ρ for Γ (see also [18]).

The origin of this notion relies on a natural generalization of quasiconformality. Given a function $Q : G \rightarrow [1, \infty]$, we say that a sense preserving homeomorphism $f : G \rightarrow \mathbb{R}^n$ is $Q(x)$ -quasiconformal if $f \in W_{\text{loc}}^{1,n}(G)$ and

$$\max\{H_I(x, f), H_O(x, f)\} \leq Q(x) \quad \text{a.e.}$$

Any $Q(x)$ -quasiconformal mapping $f : G \rightarrow \mathbb{R}^n$ is differentiable a.e., satisfies the (N) -property, and $J(x, f) \geq 0$ a.e. If, in addition, $Q \in L_{\text{loc}}^{n-1}$, then $f^{-1} \in W_{\text{loc}}^{1,n}(G^*)$ and is differentiable a.e.; f has (N^{-1}) -property and $J(x, f) > 0$ a.e. All this follows from [4, 11, 20, 21] (cf. [18]).

Given a measurable function $Q : G \rightarrow [0, \infty]$, a homeomorphism $f : G \rightarrow \mathbb{R}^n$ is called Q -homeomorphism with respect to α -module, if

$$\mathcal{M}_\alpha(f(\mathcal{S}_k)) \leq \int_G Q(x) \rho^\alpha(x) dx \quad (4.2)$$

for every family of k -dimensional surfaces \mathcal{S}_k in G and for every admissible function ρ for \mathcal{S}_k ; $1 \leq k \leq n-1$, and such integer k is fixed.

It is the well-known fact that quasiconformal mappings preserve their n -moduli up to an absolute factor, i.e.

$$K^{\frac{k-n}{n-1}} \mathcal{M}(\mathcal{S}_k) \leq \mathcal{M}(f(\mathcal{S}_k)) \leq K^{\frac{n-k}{n-1}} \mathcal{M}(\mathcal{S}_k) \quad (4.3)$$

(a quasiinvariance of n -module). For conformal mappings the n -module becomes an invariant. Observe that the inequality (4.1) is a natural generalization of the right-hand side in (4.3) for the curve families. Note also that the integral in (4.2) can be interpreted as a weighted module (cf. [1, 5, 25]).

We now restrict ourselves by the case $k = 1$, which corresponds to the curve families. For Q -homeomorphisms with $Q \in L_{\text{loc}}^1$, the differentiability a.e. and ACL-property were established in [22]. It is also known that Q -homeomorphisms satisfy the (N^{-1}) -property (see [18]).

To establish the differential properties of Q -homeomorphisms with respect to α -moduli, we first consider some set functions. Let Φ be a finite

nonnegative function in G defined for open subsets E of G satisfying

$$\sum_{k=1}^m \Phi(E_k) \leq \Phi(E)$$

for any finite collection $\{E_k\}_{k=1}^m$ of nonintersecting open sets $E_k \subset E$. Denote the class of all such set functions Φ by \mathcal{F} .

The *upper* and *lower* derivatives of a set function $\Phi \in \mathcal{F}$ at a point $x \in G$ are defined by

$$\overline{\Phi}'(x) = \lim_{h \rightarrow 0} \sup_{d(Q) < h} \frac{\Phi(Q)}{mQ}, \quad \underline{\Phi}'(x) = \lim_{h \rightarrow 0} \inf_{d(Q) < h} \frac{\Phi(Q)}{mQ},$$

where Q ranges over all open cubes or open balls such that $x \in Q \subset G$. Due to [20], these derivatives have the following properties: $\overline{\Phi}'(x)$ and $\underline{\Phi}'(x)$ are Borel's functions; $\overline{\Phi}'(x) = \underline{\Phi}'(x) < \infty$ a.e. in G ; and for each open set $V \subset G$,

$$\int_V \overline{\Phi}'(x) dx \leq \Phi(V).$$

Theorem 4.1. *Let $f : G \rightarrow G^*$ be an Q -homeomorphism with respect to α -module with $Q \in L_{\text{loc}}^1(G)$ and $\alpha > n - 1$. Then f is ACL-homeomorphism which is differentiable a.e. in G .*

For the proof of Theorem 4.1 we refer to [8] (cf. [22] for $\alpha = n$).

Corollary 4.1. *Under the assumptions of Theorem 4.1, any Q -homeomorphism with respect to α -module belongs to $W_{\text{loc}}^{1,1}$.*

The following theorem implies the upper estimates for the maximal stretching and Jacobian of f .

Theorem 4.2. *Let G and G^* be two bounded domains in \mathbb{R}^n , $n \geq 2$, and $f : G \rightarrow G^*$ be a sense preserving Q -homeomorphism with respect to α -module, $n - 1 < \alpha < n$, so that $Q \in L_{\text{loc}}^1$. Then*

$$\|f'(x)\| \leq \lambda_{n,\alpha} Q^{\frac{1}{n-\alpha}}(x) \quad \text{a.e.} \quad (4.4)$$

and

$$J(x, f) \leq \gamma_{n,\alpha} Q^{\frac{n}{n-\alpha}}(x) \quad \text{a.e.}, \quad (4.5)$$

where $\lambda_{n,\alpha}$ and $\gamma_{n,\alpha}$ are a positive constants depending only on n and α .

Proof. As well known (see [20]),

$$\limsup_{r \rightarrow 0} \frac{mf(B(x, r))}{mB(x, r)} < \infty \quad \text{and} \quad \limsup_{r \rightarrow 0} \frac{1}{mB(x, r)} \int_{B(x, r)} Q(x) dx < \infty$$

a.e. in G . Let $A = A(x_0, \varepsilon_1, \varepsilon_2) = \{x \mid \varepsilon_1 < |x - x_0| < \varepsilon_2\}$ be a spherical ring centered at $x_0 \in G$, with radii ε_1 and ε_2 , $0 < \varepsilon_1 < \varepsilon_2$, satisfying $A(x_0, \varepsilon_1, \varepsilon_2) \subset G$, choosing x_0 so that the above limits are simultaneously finite at x_0 . Then $(f(B(x_0, \varepsilon_2)), f(\overline{B}(x_0, \varepsilon_1)))$ is a ring condenser in G^* and in accordance with (3.2),

$$\text{cap}_\alpha (f(B(x_0, \varepsilon_2)), f(\overline{B}(x_0, \varepsilon_1))) = \mathcal{M}_\alpha(\Delta(\partial f(B(x_0, \varepsilon_2)), \partial f(B(x_0, \varepsilon_1))); f(A)).$$

Since f is homeomorphic

$$\Delta(\partial f(B(x_0, \varepsilon_2)), \partial f(B(x_0, \varepsilon_1))); f(A) = f(\Delta(\partial B(x_0, \varepsilon_2), \partial B(x_0, \varepsilon_1)); A).$$

Pick the admissible function

$$\rho(x) = \begin{cases} \frac{1}{\varepsilon_2 - \varepsilon_1}, & x \in A \\ 0, & x \notin A. \end{cases}$$

Since f is a Q -homeomorphism with respect to α -module,

$$\text{cap}_\alpha (f(B(x_0, \varepsilon_2)), f(\overline{B}(x_0, \varepsilon_1))) \leq \frac{1}{(\varepsilon_2 - \varepsilon_1)^\alpha} \int_{A(x_0, \varepsilon_1, \varepsilon_2)} Q(x) dx. \quad (4.6)$$

Choosing $\varepsilon_1 = 2\varepsilon$ and $\varepsilon_2 = 4\varepsilon$, we get

$$\text{cap}_\alpha (f(B(x_0, 4\varepsilon)), f(\overline{B}(x_0, 2\varepsilon))) \leq \frac{1}{(2\varepsilon)^\alpha} \int_{B(x_0, 4\varepsilon)} Q(x) dx. \quad (4.7)$$

On the other hand, the inequality (3.3) implies

$$\text{cap}_\alpha (fB(x_0, 4\varepsilon), f(\overline{B}(x_0, 2\varepsilon))) \geq C_{n, \alpha} [mf(B(x_0, 2\varepsilon))]^{\frac{n-\alpha}{n}}, \quad (4.8)$$

where the constant $C_{n, \alpha}$ depends only on the dimension n and α .

Combining (4.7) and (4.8) yields

$$\frac{mf(B(x_0, 2\varepsilon))}{mB(x_0, 2\varepsilon)} \leq c_{n, \alpha} \left[\int_{B(x_0, 4\varepsilon)} Q(x) dx \right]^{\frac{n}{n-\alpha}}, \quad (4.9)$$

where $c_{n, \alpha}$ also depends on n and α . As $\varepsilon \rightarrow 0$, the estimate (4.5) follows.

Now choosing $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = 2\varepsilon$ in (4.6), we obtain

$$\text{cap}_\alpha (f(B(x_0, 2\varepsilon)), f(\overline{B}(x_0, \varepsilon))) \leq \frac{1}{\varepsilon^\alpha} \int_{B(x_0, 2\varepsilon)} Q(x) dx,$$

and since by (3.4),

$$\text{cap}_\alpha (f(B(x_0, 2\varepsilon)), f(\overline{B}(x_0, \varepsilon))) \geq C_{n,\alpha} \frac{[d(f(B(x_0, \varepsilon)))]^{\frac{\alpha}{n-1}}}{[mf(B(x_0, 2\varepsilon))]^{\frac{1-n+\alpha}{n-1}}},$$

we have

$$\frac{d(fB(x_0, \varepsilon))}{\varepsilon} \leq \gamma_{n,\alpha} \left(\frac{mf(B(x_0, 2\varepsilon))}{mB(x_0, 2\varepsilon)} \right)^{\frac{1-n+\alpha}{\alpha}} \left(\int_{B(x_0, 2\varepsilon)} Q(x) dx \right)^{\frac{n-1}{\alpha}}, \quad (4.10)$$

where $\gamma_{n,\alpha}$ is a positive constant depending on n and α .

The inequalities (4.10) and (4.9) yield

$$\frac{d(f(B(x_0, \varepsilon)))}{\varepsilon} \leq \lambda_{n,\alpha} \left(\int_{B(x_0, 4\varepsilon)} Q(x) dx \right)^{\frac{n(1-n+\alpha)}{\alpha(n-\alpha)}} \left[\int_{B(x_0, 2\varepsilon)} Q(x) dx \right]^{\frac{n-1}{\alpha}}.$$

Finally, letting $\varepsilon \rightarrow 0$, one derives

$$\limsup_{x \rightarrow x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq \limsup_{\varepsilon \rightarrow 0} \frac{d(f(B(x_0, \varepsilon)))}{\varepsilon} \leq \lambda_{n,\alpha} Q^{\frac{1}{n-\alpha}}(x_0),$$

where $\lambda_{n,\alpha}$ is a positive constant which depends only on n and α . Thus (4.4) follows. \square

Corollary 4.2. *Let G and G^* be two domains in \mathbb{R}^n , $n \geq 2$, and let $f : G \rightarrow G^*$ be a Q -homeomorphism with respect to α -module, $n - 1 < \alpha < n$. Assume that $Q(x) \in L_{\text{loc}}^{\frac{s}{n-\alpha}}$ with $s > n - \alpha$. Then $f \in W_{\text{loc}}^{1,s}$.*

Indeed for any compact set $V \subset G$,

$$\int_V \|f'(x)\|^s dx \leq \lambda_{n,\alpha}^s \int_V Q^{\frac{s}{n-\alpha}}(x) dx < \infty.$$

As well known, every $W_{\text{loc}}^{1,n}$ -homeomorphism possesses the (N) -property; thus we have

Corollary 4.3. *If $Q \in L_{\text{loc}}^{\frac{n}{n-\alpha}}$ then f satisfies (N) -property.*

Corollary 4.4. *Let $f : G \rightarrow G^*$ be a Q -homeomorphism with respect to α -module such that $Q(x) \in L_{\text{loc}}^{\frac{n}{n-\alpha}}$, $n - 1 < \alpha < n$. Then*

$$mf(E) \leq \gamma_{n,\alpha} \int_E Q^{\frac{n}{n-\alpha}}(x) dx.$$

Proof. Since $Q(x) \in L_{\text{loc}}^{\frac{n}{n-\alpha}}(G)$, f satisfies Lusin's (N)-property and

$$mf(E) = \int_E J(x, f) dx \leq \gamma_{n,\alpha} \int_E Q^{\frac{n}{n-\alpha}}(x) dx.$$

□

The following result shows that the dilatation $H_{I,\alpha}$ is dominated a.e. in G by the upper derivative of the set function

$$\Psi(V) = \int_V Q(x) dx,$$

where V is an open subset of G .

Theorem 4.3. *Let $f : G \rightarrow G^*$ be a Q -homeomorphism with respect to α -module, $\alpha > n - 1$, with $Q \in L_{\text{loc}}^1(G)$ such that $J(x, f) \neq 0$ a.e. in G . Then*

$$H_{I,\alpha}(x, f) \leq \overline{\Psi}'(x) \quad \text{a.e. in } G.$$

Proof. Let $a \in G$ be an arbitrary point where f is differentiable at a , with $J(a, f) \neq 0$ and $\overline{\Psi}'(a) \neq 0$. The image of the unit ball under the linear mapping $f'(a)$ is an ellipsoid $E(f)$ with semi-axes $\lambda_1, \lambda_2, \dots, \lambda_n$ ordered by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$. Preceding f , if necessary, by a rotation and a reflection, one reduces to the case $f(a) = a = 0$ and $|f'(0)e_i| = \lambda_i$, $i = 1, \dots, n$; here e_ν denotes the ν th unit basis vector.

For every $t > 0$, let \mathcal{R} be the ring domain obtained from n -dimensional interval

$$I_n = \{x \mid |x_i| < r(t\lambda_i + 1), i = 1, \dots, n-1, |x_n| < rt\lambda_n\},$$

by deleting the points of $(n-1)$ -dimensional interval

$$\Pi_{n-1}(0, r) = \{x : |x_i| \leq r, i = 1, \dots, n-1, x_n = 0\}.$$

We choose $r > 0$ so that $\overline{\mathcal{R}} \subset G$ and will show that

$$\frac{\lambda_1 \cdot \dots \cdot \lambda_n}{\lambda_n^\alpha} \leq \overline{\Psi}'(0). \quad (4.11)$$

Indeed, the inequality

$$\mathcal{M}_\alpha(f(\Gamma_{\mathcal{R}})) \leq \int_{\mathcal{R}} Q(x) \rho^\alpha(x) dx,$$

together with the following estimate (see, e.g. [14])

$$\mathcal{M}_\alpha(\Gamma_{\mathcal{R}}) \geq \frac{(\inf_{\Sigma} m_{n-1}\Sigma)^\alpha}{(m\mathcal{R})^{\alpha-1}},$$

gives

$$\frac{\left(\inf_{\Sigma^*} m_{n-1} \Sigma^*\right)^\alpha}{(mf(\mathcal{R}))^{\alpha-1}} \leq \frac{1}{(\text{dist}(F_0, F_1))^\alpha} \int_{\mathcal{R}} Q(x) dx,$$

where the infimum is taken over all surfaces Σ^* which separate $f(C_0)$ and $f(C_1)$ in $f(\mathcal{R})$.

Fix $0 < \varepsilon < \lambda_n$ and choose $r > 0$ so small that

$$mf(I_n) \leq (J(0, f) + \varepsilon)mI_n \quad \text{and} \quad |f(x) - f'(0)x| < \varepsilon r.$$

Since

$$m\mathcal{R} = mI_n = 2^n r^n t \lambda_n (t\lambda_1 + 1) \cdot \dots \cdot (t\lambda_{n-1} + 1),$$

and

$$\inf_{\Sigma^*} m_{n-1} \Sigma^* \geq 2m_{n-1} \tilde{\Pi}_{n-1}(0, r) = 2^n r^{n-1} (\lambda_1 - \varepsilon) \cdot \dots \cdot (\lambda_{n-1} - \varepsilon),$$

we have

$$\frac{[2^n r^{n-1} (\lambda_1 - \varepsilon) \cdot \dots \cdot (\lambda_{n-1} - \varepsilon)]^\alpha}{[(J(0, f) + \varepsilon)mA]^{\alpha-1}} \leq \frac{1}{(rt\lambda_n)^\alpha} \int_{\mathcal{R}} Q(x) dx,$$

where $\tilde{\Pi}_{n-1}(0, r) = \{y \mid |y_i| \leq r\lambda_i - \varepsilon r, i = 1, \dots, n-1, y_n = 0\}$. Letting $t \rightarrow 0$ and thereafter $r \rightarrow 0$, we get

$$\frac{[(\lambda_1 - \varepsilon) \cdot \dots \cdot (\lambda_{n-1} - \varepsilon)]^\alpha}{[J(0, f) + \varepsilon]^{\alpha-1}} \leq \bar{\Psi}'(0),$$

which implies (4.11) as $\varepsilon \rightarrow 0$. Hence $H_{I,\alpha}(x, f) \leq \bar{\Psi}'(x)$ for almost all $x \in \mathcal{R}$. \square

Remark 4.1. If one omits the restriction $\alpha > n - 1$ (i.e. for $1 \leq \alpha \leq n - 1$), Theorem 4.3 can be proved assuming additionally that the Q -homeomorphisms f with respect to α -module are differentiable a.e.

Remark 4.2. The inequality in Theorem 4.3 can be replaced by $H_{I,\alpha}(x, f) \leq Q(x)$ a.e.

5. Ring Q -homeomorphisms and their properties

Recall some necessary notions. Let $E, F \subseteq \mathbb{R}^n$ be arbitrary domains. Denote by $\Delta(E, F, G)$ the family of all curves $\gamma : [a, b] \rightarrow \mathbb{R}^n$, which join E and F in G , i.e. $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$.

Set $d_0 = \text{dist}(x_0, \partial G)$ and let $Q : G \rightarrow [0, \infty]$ be a Lebesgue measurable function. Denote

$$A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n \mid r_1 < |x - x_0| < r_2\},$$

and

$$S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n \mid |x - x_0| = r_i\}, \quad i = 1, 2. \quad (5.1)$$

We say that a homeomorphism $f : G \rightarrow \mathbb{R}^n$ is *the ring Q -homeomorphism with respect to p -module at the point $x_0 \in G$* , ($1 < p \leq n$) if the inequality

$$\mathcal{M}_p(\Delta(f(S_1), f(S_2), f(G))) \leq \int_A Q(x) \cdot \eta^p(|x - x_0|) dx \quad (5.2)$$

is fulfilled for any ring $A = A(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0$ and for every measurable function $\eta : (r_1, r_2) \rightarrow [0, \infty]$, satisfying

$$\int_{r_1}^{r_2} \eta(r) dr \geq 1. \quad (5.3)$$

The homeomorphism $f : G \rightarrow \mathbb{R}^n$ is the *ring Q -homeomorphism with respect to p -module in the domain G* , if inequality (5.2) holds for all points $x_0 \in G$. The properties of the ring Q -homeomorphisms for $p = n$ are studied in [24].

The ring Q -homeomorphisms are defined in fact locally and contain as a proper subclass of Q -homeomorphisms (see [18]). A necessary and sufficient condition for homeomorphisms to be ring Q -homeomorphisms with respect to p -module at a point given in [23], asserts:

Proposition 5.1. *Let G be a bounded domain in \mathbb{R}^n , $n \geq 2$ and let $Q : G \rightarrow [0, \infty]$ belong to L^1_{loc} . A homeomorphism $f : G \rightarrow \mathbb{R}^n$ is a ring Q -homeomorphism with respect to p -module at $x_0 \in G$ if and only if for any $0 < r_1 < r_2 < d_0 = \text{dist}(x_0, \partial G)$,*

$$\mathcal{M}_p(\Delta(f(S_1), f(S_2), f(G))) \leq \frac{\omega_{n-1}}{I^{p-1}},$$

where S_1 and S_2 are the spheres defined in (5.1)

$$I = I(x_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)},$$

and $q_{x_0}(r)$ is the mean value of Q over $|x - x_0| = r$. Note that the infimum in the right-hand side of (5.2) over all admissible η satisfying (5.3) is attained only for the function

$$\eta_0(r) = \frac{1}{I r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)}.$$

6. Lower Q -homeomorphisms and their module bounds

Let G and G^* be two bounded domains in \mathbb{R}^n , $n \geq 2$ and $x_0 \in G$. Given a Lebesgue measurable function $Q : G \rightarrow [0, \infty]$, a homeomorphism $f : G \rightarrow G^*$ is called the *lower Q -homeomorphism with respect to p -module at x_0* if

$$\mathcal{M}_p(f(\Sigma_\varepsilon)) \geq \inf_{\rho \in \text{expadm } \Sigma_\varepsilon} \int_{D_\varepsilon(x_0)} \frac{\rho^p(x)}{Q(x)} dx, \quad (6.1)$$

where

$$G_\varepsilon(x_0) = G \cap \{x \in \mathbb{R}^n \mid \varepsilon < |x - x_0| < \varepsilon_0\}, \quad 0 < \varepsilon < \varepsilon_0, \quad 0 < \varepsilon_0 < \sup_{x \in G} |x - x_0|,$$

and Σ_ε denotes the family of all pieces of spheres centered at x_0 of radii r , $\varepsilon < r < \varepsilon_0$, located in G .

In this section, we provide a necessary and sufficient condition for homeomorphisms to be lower Q -homeomorphisms with respect to p -module. The case $p = n$ is considered in [18].

Theorem 6.1. *Let G be a domain in \mathbb{R}^n , $n \geq 2$, $x_0 \in \overline{G}$, and let $Q : G \rightarrow [0, \infty]$ be a measurable function. A homeomorphism $f : G \rightarrow \mathbb{R}^n$ is a lower Q -homeomorphism at x_0 with respect to p -module for $p > n - 1$ if and only if the following inequality*

$$\mathcal{M}_p(f(\Sigma_\varepsilon)) \geq \int_\varepsilon^{\varepsilon_0} \frac{dr}{\|Q\|_s(r)} \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \varepsilon_0 \in (0, d_0),$$

holds, where $s = \frac{n-1}{p-n+1}$,

$$d_0 = \sup_{x \in D} |x - x_0|,$$

Σ_ε is the family of all intersections of the spheres $S(x_0, r) = \{x \in \mathbb{R}^n \mid |x - x_0| = r\}$, $r \in (\varepsilon, \varepsilon_0)$ with G and

$$\|Q\|_s(r) = \left(\int_{G(x_0, r)} Q^s(x) d\sigma_{n-1} \right)^{\frac{1}{s}}$$

here $G(x_0, r) = \{x \in G \mid |x - x_0| = r\} = G \cap S(x_0, r)$. The infimum in (6.1) is attained only on the functions

$$\varrho_0(x) = \frac{Q(x)}{\|Q\|_s(r)}.$$

Proof. For any $\varrho \in \text{extadm } \Sigma_\varepsilon$ the function

$$A_\varrho(r) := \int_{G(x_0, r)} \varrho^{n-1}(x) d\sigma_{n-1} \neq 0$$

is measurable on $(\varepsilon, \varepsilon_0)$. It is admissible if $A_\varrho(r) \geq 1$. Assuming that $A_\varrho(r) \equiv 1$, we obtain

$$\inf_{\varrho \in \text{extadm } \Sigma_\varepsilon} \int_{G_\varepsilon(x_0)} \frac{\varrho^p(x)}{Q(x)} dx = \int_\varepsilon^{\varepsilon_0} \left(\inf_{\psi \in I(r)} \int_{G(x_0, r)} \frac{\psi^q(x)}{Q(x)} d\sigma_{n-1} \right) dr,$$

where $q = p/(n-1) > 1$ and $I(r)$ denotes the set of all measurable functions ψ on the surface $G(x_0, r)$ satisfying

$$\int_{G(x_0, r)} \psi(x) d\sigma_{n-1} = 1.$$

Thus Theorem 6.1 follows from Lemma 3.1 taking there $X = G(x_0, r)$ and μ to be the $(n-1)$ -dimensional area on $G(x_0, r)$, and $\varphi(x) = 1/Q(x)$ on $G(x_0, r)$, and $q = p/(n-1) > 1$. This completes the proof. \square

7. Connection between the ring and lower Q -homeomorphisms

In this sections we establish the relationship between the ring and lower Q -homeomorphisms with respect to p -module.

Theorem 7.1. *Every lower Q -homeomorphism with respect to p -module $f : G \rightarrow G^*$ at $x_0 \in G$, with $p > n-1$ and $Q \in L_{\text{loc}}^{\frac{n-1}{p-n+1}}$, is a ring \tilde{Q} -homeomorphism with respect to α -module at x_0 with $\tilde{Q} = Q^{\frac{n-1}{p-n+1}}$ and $\alpha = \frac{p}{p-n+1}$.*

Proof. Let $0 < r_1 < r_2 < d(x_0, \partial G)$ and $S_i = S(x_0, r_i)$, $i = 1, 2$, be from (5.1). Then taking into account the relation (3.1), we obtain

$$\mathcal{M}_\alpha(f(\Delta(S_1, S_2, G))) \leq \frac{1}{\mathcal{M}_p^{\frac{n-1}{p-n+1}}(f(\Sigma))}, \quad (7.1)$$

where $f(\Sigma) \subset \Sigma(f(S_1), f(S_2), f(G))$ and Σ denotes the family of all spheres centered at x_0 , located between S_1 and S_2 , while $\Sigma(f(S_1), f(S_2), f(G))$ consists of all $(n-1)$ -dimensional surfaces in $f(G)$, separating $f(S_1)$ and $f(S_2)$ (cf. [18]) Now directly by (7.1) and Theorem 6.1,

$$\mathcal{M}_\alpha(f(\Delta(S_1, S_2, G))) \leq \left(\int_{r_1}^{r_2} \frac{dr}{\|Q\|_{\frac{n-1}{p-n+1}}(r)} \right)^{\frac{1-n}{p-n+1}} = \frac{\omega_{n-1}}{\tilde{I}^{\frac{n-1}{p-n+1}}}, \quad (7.2)$$

where $\tilde{I} = \tilde{I}(x_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{\alpha-1}} \tilde{q}_{x_0}^{\frac{1}{\alpha-1}}(r)}$, and $\tilde{q}_{x_0}(r)$ denotes the mean value of the function \tilde{Q} over $|x - x_0| = r$. Now the assertion of the theorem follows from (7.2) and Proposition 5.1. \square

References

- [1] C. ANDREIAN CAZACU, *Some formulae on the extremal length in n -dimensional case*. Proceedings of the Romanian-Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings (Braşov, 1969), pp. 87–102. Publ. House of the Acad. of the Socialist Republic of Romania, Bucharest, 1971.
- [2] C. ANDREIAN CAZACU, *Module inequalities for quasiregular mappings*, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **2** (1976), 17-28.
- [3] C. ANDREIAN CAZACU, *On the length-area dilatation*, *Complex Var. Theory Appl.*, **50** (2005), no. 7-11, 765-776.
- [4] M. CRISTEA, *Local homeomorphisms having local ACLⁿ inverses*, *Complex Var. Elliptic Equ.*, **53** (2008), 77-99.
- [5] M. CRISTEA, *Mappings satisfying some modular inequalities*, Institute of Mathematics of the Romanian Academy, preprint, 2011, 12 pp.
- [6] F. W. GEHRING, *Lipschitz mappings and the p -capacity of rings in n -space*, Advances in the theory of Riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969), pp.175-193, Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.
- [7] A. GOLBERG, *Homeomorphisms with finite mean dilatations*, *Contemp. Math.*, **382** (2005), 177-186.
- [8] A. GOLBERG, *Differential properties of (α, Q) -homeomorphisms*, *Further progress in analysis*, 218-228, World Sci. Publ., Hackensack, N.J., 2009.
- [9] A. GOLBERG, *Homeomorphisms with integrally restricted moduli*, *Contemp. Math.*, **553** (2011), 83-98.
- [10] V. GOL'DSHTEIN and YU.G. RESHETNYAK, *Quasiconformal mappings and Sobolev spaces*, Kluwer Academic Publishers Group, Dordrecht, 1990.
- [11] J. HEINONEN and P. KOSKELA, *Sobolev mappings with integrable dilatation*, *Arch. Rational Mech. Anal.*, **125** (1993), 81-97.
- [12] J. HESSE, *p -extremal length and p -capacity equality*, *Ark. Mat.*, **13** (1975), 131-144.
- [13] D. KOVTONYK and V. RYAZANOV, *On the theory of mappings with finite area distortion*, *J. Anal. Math.*, **104** (2008), 291-306.
- [14] V. I. KRUGLIKOV, *Capacities of condensers and quasiconformal in the mean mappings in space*, *Mat. Sb.*, **130** (1986), 185-206.
- [15] S. L. KRUSHKAL, *Quasiconformal mappings and Riemann surfaces*. V. H. Winston & Sons, Washington, D.C.; John Wiley & Sons, New York-Toronto, Ont.-London, 1979.
- [16] O. MARTIO, S. RICKMAN and J. VÄISÄLÄ, *Definitions for quasiregular mappings*, *Ann. Acad. Sci. Fenn. Ser. A I*, **448** (1969), 1-40.
- [17] O. MARTIO, V. RYAZANOV, U. SREBRO and E. YAKUBOV, *Q -homeomorphisms*, *Contemp. Math.*, **364** (2004), 193-203.
- [18] O. MARTIO, V. RYAZANOV, U. SREBRO and E. YAKUBOV, *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics, XII, 2009.

- [19] V. MAZ'YA, Lectures on isoperimetric and isocapacitary inequalities in the theory of Sobolev spaces, *Contemp. Math.*, **338** (2003), 307-340.
- [20] T. RADO and P. REICHELDERFER, *Continuous Transformations in Analysis*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.
- [21] YU.G. RESHETNYAK, *Space Mappings with Bounded Distortion*, Trans. of Mathematical Monographs, Amer. Math. Soc., vol. 73, 1989.
- [22] R. SALIMOV, ACL and differentiability of Q -homeomorphisms, *Ann. Acad. Sci. Fenn. Math.*, **33** (2008), 295-301.
- [23] R. SALIMOV, On finitely lipschitz space mappings, *Sib. Èlektron. Mat. Izv.*, **8** (2011), 284-295.
- [24] R. SALIMOV and E. SEVOST'YANOV, Theory of ring Q -mappings and geometric function theory, *Sb. Math.*, **201** (2010), 909-934.
- [25] P. M. TAMRAZOV, Moduli and extremal metrics in nonorientable and twisted Riemannian manifolds, *Ukrainian Math. J.*, **50** (1998), 1586-1598.
- [26] W. P. ZIEMER, Extremal length and p -capacity, *Michigan Math. J.*, **16** (1969), 43-51.

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