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Optimality and duality for nondifferentiable minimax fractional optimization problems with generalized invexity

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Abstract - We establish Karush-Kuhn-Tucker sufficient optimality conditions for a nondifferentiable minimax fractional optimization problem, in which numerator and denominator of each term consists of support function, under the assumptions of (V, ρ, σ)-type I invex functions. Mond-Weir type weak and strong duality theorems are also obtained under the aforesaid assumptions.

Key words and phrases : sufficient optimality conditions, duality theorems, generalized invexity, minimax fractional programming problems.

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1. Introduction

The concept of duality, convexity and invexity in fractional programming has received special attention of researchers in solving different real life problems and mathematical models that require the relative comparison of two magnitudes. The popularity of the fractional programming lies in the fact that although the objective function is nondifferentiable, a simple formulation of the dual may be given. Later, Mishra et al. [9], Chinchuluun et al. [1] and Huang et al. [7] have dealt with many useful optimality and duality results for nondifferentiable fractional programming problems. There are several chapters devoted to this topic by Mishra and Giorgio [8], Clarke [2] and Craven [3].

In the recent past, Kuk et al. [6] have introduced the concept of (V, ρ)-invexity, which is generalization of the V-invexity for vector valued functions and derived the generalized Karush-Kuhn-Tucker optimality conditions as well as weak and strong duality for nonsmooth multiobjective fractional programs. Later, Kim et al. [4] extended their results in presence of support functions. Very recently, Kim et al. [5] have introduced the assumption of (V, ρ)-invexity for the following generalized nondifferentiable fractional programming problem (GFP):

$$\text{Minimize} \quad \max \left\{ \frac{f_i(x) + s(x|C_i)}{g_i(x) - s(x|D_i)} \mid i = 1, \ldots, p \right\}$$

$$\text{subject to:} \qquad h_j(x) \leq 0,\ j = 1, \ldots m,$$

where $f := (f_1, \ldots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$, $g := (g_1, \ldots, g_p) : \mathbb{R}^n \to \mathbb{R}^p$ and $h := (h_1, \ldots, h_m) : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable and for each $i = 1, \ldots, p$, C_i and D_i are compact convex sets of $\mathbb{R}^n$.

In this paper, motivated by Kim et al. (see [5]), we introduce (V, ρ, σ)-type I invex function to derive the Karush-Kuhn-Tucker sufficient optimality and Mond-Weir type weak and strong duality theorems for a generalized nondifferentiable minimax fractional optimization problem (GFP), in which numerator and denominator of each term consists of support function, and a constraint set defined by differentiable functions.

2. Preliminaries and definitions

In this paper, we consider the following nondifferentiable multiobjective fractional programming problem

$$\text{(GFP)} \qquad \text{Minimize} \quad \max \left\{ \frac{f_i(x) + s(x|C_i)}{g_i(x) - s(x|D_i)} \,|\, i = 1, \ldots, p \right\}$$

$$\text{subject to:} \qquad h_j(x) \leq 0,\ j = 1, \ldots m,$$

where $f := (f_1, \ldots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$, $g := (g_1, \ldots, g_p) : \mathbb{R}^n \to \mathbb{R}^p$ and $h := (h_1, \ldots, h_m) : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable. We assume that $g_i(x) - s(x|D_i) > 0$, $i = 1, \ldots, p$. For each $i = 1, \ldots, p$, C_i and D_i are compact convex sets of $\mathbb{R}^n$ and define the support functions with respect to C_i and D_i as follows:

$$s(x|C_i) = \max\{\langle x, y_i \rangle \,|\, y_i \in C_i\}$$

and

$$s(x|D_i) = \max\{\langle x, y_i \rangle \,|\, y_i \in D_i\}.$$

Further denote $I(x) = \{j \,|\, h_j(x) = 0\}$, for any $x \in \mathbb{R}^n$. Let $k_i(x) = s(x|C_i)$ and $\tilde{k}_i(x) = s(x|D_i)$, $i = 1, \ldots, p$. Hence, $k_i(x)$ and $\tilde{k}_i(x)$ are convex functions. Choose $\omega_i \in \partial k_i(x)$ and $\tilde{\omega}_i \in \partial \tilde{k}_i(x)$ such that

$$\partial k_i(x) = \{\omega_i \in C_i \,|\, \langle \omega_i, x \rangle = s(x|C_i)\}$$

and

$$\partial \tilde{k}_i(x) = \{\tilde{\omega}_i \in D_i \,|\, \langle \tilde{\omega}_i, x \rangle = s(x|D_i)\},$$

where ∂k_i and $\partial \tilde{k}_i$ are the subdifferential of k_i and $\tilde{k}_i$ respectively. Further, let

$$S = \{x \in \mathbb{R}^n \,|\, h_j(x) \leq 0, j = 1, \ldots, m\}.$$

Definition 2.1. *A vector valued function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be η-invex at $u \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n$ and for all $i = 1, \ldots, p$, one has*

$$f_i(x) - f_i(u) \geq \nabla f_i(u)\eta(x, u).$$

Definition 2.2. *A vector valued function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be (V, ρ)-invex at $u \in \mathbb{R}^n$ with respect to the functions η and $\theta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if there exist $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ and $\rho_i \in \mathbb{R}$, $i = 1, \ldots, p$, such that for any $x \in \mathbb{R}^n$ and for all $i = 1, \ldots, p$ it holds*

$$\alpha_i(x, u)\,[f_i(x) - f_i(u)] \geq \nabla f_i(u)\eta(x, u) + \rho_i\|\theta_i(x, u)\|^2.$$

The following Theorem from [5] will be needed in the sequel:

Theorem 2.1. *Assume that f and g are vector valued differentiable functions defined on $\mathbb{R}^n$ and $f(x) + \langle \omega, x\rangle \geq 0, g(x) - \langle \tilde{\omega}, x\rangle > 0$ for all $x \in \mathbb{R}^n$. If $f(\cdot) + \langle \omega, \cdot\rangle$ and $-g(\cdot) + \langle \tilde{\omega}, \cdot\rangle$ are (V, ρ)-invex at $u \in \mathbb{R}^n$ with respect to the functions η, θ_i and α_i, $i = 1, \ldots, p$, then $\frac{f(\cdot)+\langle\omega,\cdot\rangle}{g(\cdot)-\langle\tilde{\omega},\cdot\rangle}$ is (V, ρ)- invex at $u \in \mathbb{R}^n$ with respect to the functions η, $\bar{\theta}_i$ and $\bar{\alpha}_i$, $i = 1, \ldots, p$, where*

$$\bar{\alpha}_i(x, u) = \frac{g_i(x) - \langle\tilde{\omega}_i, x\rangle}{g_i(u) - \langle\tilde{\omega}_i, u\rangle}\alpha_i(x, u)$$

and

$$\bar{\theta}_i(x, u) = \left(\frac{1}{g_i(u) - \langle\tilde{\omega}_i, u\rangle}\right)^{\frac{1}{2}} \theta_i(x, u),$$

that is, for all i,

$$\bar{\alpha}_i(x, u)\left[\frac{f_i(x) + \langle\omega_i, x\rangle}{g_i(x) - \langle\tilde{\omega}_i, x\rangle} - \frac{f_i(u) + \langle\omega_i, u\rangle}{g_i(u) - \langle\tilde{\omega}_i, u\rangle}\right] \geq \nabla\left(\frac{f_i(u) + \langle\omega_i, u\rangle}{g_i(u) - \langle\tilde{\omega}_i, u\rangle}\right)\eta(x, u) + \rho_i\|\bar{\theta}_i(x, u)\|^2.$$

Definition 2.3. *Let*

$$\left(\frac{f_i(x) + \langle\omega_i, x\rangle}{g_i(x) - \langle\tilde{\omega}_i, x\rangle}\right) = \phi_i(x), \quad i = 1, \ldots, p,$$

$$\langle\omega_i, x\rangle = s(x|C_i), \quad \langle\tilde{\omega}_i, x\rangle = s(x|D_i).$$

The pair (ϕ_i, h_j) is called (V, ρ_i, σ_j)-type I invex at $u \in \mathbb{R}^n$, if there exist $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$,

$$\bar{\alpha}_i(x, u) = \frac{g_i(x) - \langle\tilde{\omega}_i, x\rangle}{g_i(u) - \langle\tilde{\omega}_i, u\rangle}\alpha_i(x, u) > 0,$$

$$\bar{\theta}_i(x,u) = \left(\frac{1}{g_i(u) - \langle\tilde{\omega}_i, u\rangle}\right)^{\frac{1}{2}} \theta_i(x,u),$$

$\beta_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$, $\rho_i \in \mathbb{R}$, $i = 1, \dots, p$; $\sigma_j \in \mathbb{R}, j = 1, \dots, m$, *such that*

$$\phi_i(x) - \phi_i(u) \geq \bar{\alpha}_i(x,u)\nabla\phi_i(u)\eta(x,u) + \rho_i\|\bar{\theta}_i(x,u)\|^2$$

and

$$-h_j(u) \geq \beta_j(x,u)\nabla h_j(u)\eta(x,u) + \sigma_j\|\theta_j(x,u)|^2.$$

3. Optimality conditions

The following Kuhn-Tucker necessary optimality conditions for (GFP) from [5] will be needed in the sequel:

Theorem 3.1. (Kuhn-Tucker necessary optimality condition) *If x_0 is a solution of the problem* (GFP) *and under the assumption that one has $0 \notin \mathrm{co}\{\nabla h_j(x_0) \,|\, j = 1, \dots, m\}$, then there exist $\lambda_i \geq 0$,*

$$i \in I(x_0) := \left\{ i \,\Big|\, \max_j \frac{f_j(x_0) + s(x_0|C_j)}{g_j(x_0) - s(x_0|D_j)} = \frac{f_i(x_0) + s(x_0|C_i)}{g_i(x_0) - s(x_0|D_i)} \right\},$$

$\sum_{i=1}^p \lambda_i = 1$, $\mu_j \geq 0$, $j = 1, \dots, m$ *and* $\omega_i \in C_i, \tilde{\omega}_i \in D_i$, $i = 1, \dots, p$ *such that*

$$\sum_{i=1}^p \lambda_i \nabla \left(\frac{f_i(x_0) + \langle\omega_i, x_0\rangle}{g_i(x_0) - \langle\tilde{\omega}_i, x_0\rangle}\right) + \sum_{j=1}^m \mu_j \nabla h_j(x_0) = 0,$$

$$\langle\omega_i, x_0\rangle = s(x_0|C_i), \quad \langle\tilde{\omega}_i, x_0\rangle = s(x_0|D_i),$$

$$\sum_{j=1}^m \mu_j h_j(x_0) = 0.$$

Theorem 3.2. (Kuhn-Tucker type sufficient condition) *Suppose that there exist a feasible solution x_0 for* (GFP) *and scalars $\lambda_i > 0$, $i = 1, \dots, p$, $\sum_{i=1}^p \lambda_i = 1$, $\mu_j \geq 0$, $j = 1, \dots, m$ and $\omega_i \in C_i$, $\tilde{\omega}_i \in D_i$, $i = 1, \dots, p$, such that*

$$(i) \qquad \sum_{i=1}^p \lambda_i \nabla \left(\frac{f_i(x_0) + \langle\omega_i, x_0\rangle}{g_i(x_0) - \langle\tilde{\omega}_i, x_0\rangle}\right) + \sum_{j=1}^m \mu_j \nabla h_j(x_0) = 0,$$

$$\langle\omega_i, x_0\rangle = s(x_0|C_i), \quad \langle\tilde{\omega}_i, x_0\rangle = s(x_0|D_i),$$

$$\sum_{j=1}^m \mu_j h_j(x_0) = 0;$$

(ii) (ϕ_i, h_j) is (V, ρ_i, σ_j)-type I invex at x_0.
Then x_0 is an efficient solution for (GFP).

Proof. Hypothesis (i) implies that

$$0=\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(x_0)+\langle\omega_i,x_0\rangle}{g_i(x_0)-\langle\tilde{\omega}_i,x_0\rangle}\right)+\sum_{j=1}^{m}\mu_j\nabla h_j(x_0). \tag{3.1}$$

Since (ϕ_i,h_j) is (V,ρ_i,σ_j)-type I invex at x_0, we have, for all $x\in S$,

$$\begin{aligned}&\frac{f_i(x)+\langle\omega_i,x\rangle}{g_i(x)-\langle\tilde{\omega}_i,x\rangle}-\frac{f_i(x_0)+\langle\omega_i,x_0\rangle}{g_i(x_0)-\langle\tilde{\omega}_i,x_0\rangle}\\&\qquad\geq\bar{\alpha}_i(x,x_0)\nabla\left(\frac{f_i(x_0)+\langle\omega_i,x_0\rangle}{g_i(x_0)-\langle\tilde{\omega}_i,x_0\rangle}\right)\eta(x,x_0)+\rho_i\|\bar{\theta}_i(x,x_0)\|^2\end{aligned}$$

and

$$0=-h_j(x_0)\geq\beta_j(x,x_0)\nabla h_j(x_0)\eta(x,x_0)+\sigma_j\|\theta_j(x,x_0)\|^2.$$

By using $\bar{\alpha}_i(x,x_0)>0$, $i=1,\ldots,p$ and $\beta_j(x,x_0)>0$, $j=1,\ldots,m$ we get

$$\begin{aligned}&\frac{1}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x)+\langle\omega_i,x\rangle}{g_i(x)-\langle\tilde{\omega}_i,x\rangle}\right)-\frac{1}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x_0)+\langle\omega_i,x_0\rangle}{g_i(x_0)-\langle\tilde{\omega}_i,x_0\rangle}\right)\\&\qquad\geq\nabla\left(\frac{f_i(x_0)+\langle\omega_i,x_0\rangle}{g_i(x_0)-\langle\tilde{\omega}_i,x_0\rangle}\right)\eta(x,x_0)+\frac{\rho_i\|\bar{\theta}_i(x,x_0)\|^2}{\bar{\alpha}_i(x,x_0)}\end{aligned} \tag{3.2}$$

and

$$0\geq\nabla h_j(x_0)\eta(x,x_0)+\frac{\sigma_j\|\theta_j(x,x_0)\|^2}{\beta_j(x,x_0)}. \tag{3.3}$$

Adding (3.2) and (3.3), we get

$$\begin{aligned}&\sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x)+\langle\omega_i,x\rangle}{g_i(x)-\langle\tilde{\omega}_i,x\rangle}\right)-\sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x_0)+\langle\omega_i,x_0\rangle}{g_i(x_0)-\langle\tilde{\omega}_i,x_0\rangle}\right)\\&\qquad\geq\left[\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(x_0)+\langle\omega_i,x_0\rangle}{g_i(x_0)-\langle\tilde{\omega}_i,x_0\rangle}\right)+\sum_{j=1}^{m}\mu_j\nabla h_j(x_0)\right]\eta(x,x_0)\\&\qquad\quad+\sum_{i=1}^{p}\lambda_i\frac{\rho_i\|\bar{\theta}_i(x,x_0)\|^2}{\bar{\alpha}_i(x,x_0)}+\sum_{j=1}^{m}\mu_j\frac{\sigma_j\|\theta_j(x,x_0)\|^2}{\beta_j(x,x_0)}.\end{aligned}$$

Using (3.1), we have

$$\begin{aligned}&\sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x)+\langle\omega_i,x\rangle}{g_i(x)-\langle\tilde{\omega}_i,x\rangle}\right)-\sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x_0)+\langle\omega_i,x_0\rangle}{g_i(x_0)-\langle\tilde{\omega}_i,x_0\rangle}\right)\\&\qquad\geq\sum_{i=1}^{p}\lambda_i\frac{\rho_i\|\bar{\theta}_i(x,x_0)\|^2}{\bar{\alpha}_i(x,x_0)}+\sum_{j=1}^{m}\mu_j\frac{\sigma_j\|\theta_j(x,x_0)\|^2}{\beta_j(x,x_0)}.\end{aligned}$$

As $\sum_{i=1}^{p} \lambda_i \frac{\rho_i \|\bar{\theta}_i(x, x_0)\|^2}{\bar{\alpha}_i(x, x_0)} \geq 0$ and $\sum_{j=1}^{m} \mu_j \frac{\sigma_j \|\theta_j(x, x_0)\|^2}{\beta_j(x, x_0)} \geq 0$, we get

$$\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x, x_0)} \left(\frac{f_i(x) + \langle \omega_i, x \rangle}{g_i(x) - \langle \tilde{\omega}_i, x \rangle} \right) - \sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x, x_0)} \left(\frac{f_i(x_0) + \langle \omega_i, x_0 \rangle}{g_i(x_0) - \langle \tilde{\omega}_i, x_0 \rangle} \right) \geq 0.$$

We thus have

$$\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x, x_0)} \left(\frac{f_i(x) + \langle \omega_i, x \rangle}{g_i(x) - \langle \tilde{\omega}_i, x \rangle} \right) \geq \sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x, x_0)} \left(\frac{f_i(x_0) + \langle \omega_i, x_0 \rangle}{g_i(x_0) - \langle \tilde{\omega}_i, x_0 \rangle} \right). \tag{3.4}$$

Suppose that x_0 is not an efficient solution for (GFP), then there exist a feasible solution x for (GFP) and an index r such that $\phi_i(x) \leq \phi_i(x_0)$ for any $i \neq r$ and $\phi_r(x) < \phi_r(x_0)$, where

$$\phi_i(x) = \frac{f_i(x) + \langle \omega_i, x \rangle}{g_i(x) - \langle \tilde{\omega}_i, x \rangle} \quad \text{for any } i.$$

Since $\lambda_i > 0$ and $\bar{\alpha}_i(x, x_0) > 0$, $i = 1, \ldots, p$, we have

$$\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x, x_0)} \phi_i(x) < \sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x, x_0)} \phi_i(x_0).$$

It follows that one has

$$\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x, x_0)} \left(\frac{f_i(x) + \langle \omega_i, x \rangle}{g_i(x) - \langle \tilde{\omega}_i, x \rangle} \right) < \sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x, x_0)} \left(\frac{f_i(x_0) + \langle \omega_i, x_0 \rangle}{g_i(x_0) - \langle \tilde{\omega}_i, x_0 \rangle} \right),$$

which contradicts the inequalities (3.4) and hence x_0 is an efficient solution for (GFP). □

4. Mond-Weir type duality

We now consider the following Mond-Weir type dual for (GFP).

$$\text{(DGFP)} \qquad \text{Maximize} \quad \max \left\{ \frac{f_i(u) + s(u|C_i)}{g_i(u) - s(u|D_i)} \,|\, i = 1, \ldots, p \right\}$$

$$\text{subject to:} \qquad \sum_{i=1}^{p} \lambda_i \nabla \left(\frac{f_i(u) + \langle \omega_i, u \rangle}{g_i(u) - \langle \tilde{\omega}_i, u \rangle} \right) + \sum_{j=1}^{m} \mu_j \nabla h_j(u) = 0, \tag{4.1}$$

$$\omega_i \in C_i,\ \langle \omega_i, u \rangle = s(u|C_i),\ \tilde{\omega}_i \in D_i,\ \langle \tilde{\omega}_i, u \rangle = s(u|D_i), i = 1, \ldots, m,$$

$$\lambda_i > 0, i = 1, \ldots, p,\ \sum_{i=1}^{p} \lambda_i = 1, \quad \mu_j \geq 0, j = 1, \ldots, m,\ \sum_{j=1}^{m} \mu_j h_j(u) = 0.$$

Theorem 4.1. (Weak duality) *Let x be a feasible solution for* (GFP) *and let $(u, \lambda, \mu, \omega, \tilde{\omega})$ be feasible for* (DGFP) *such that (ϕ_i, h_j) is (V, ρ_i, σ_j)-type I invex at u. Then the following cannot hold*

$$\left(\frac{f_i(x) + s(x|C_i)}{g_i(x) - s(x|D_i)} \right) < \left(\frac{f_i(u) + s(u|C_i)}{g_i(u) - s(u|D_i)} \right). \tag{4.2}$$

Proof. Suppose that (4.2) holds, that is

$$\left(\frac{f_i(x) + \langle \omega_i, x \rangle}{g_i(x) - \langle \tilde{\omega}_i, x \rangle} \right) < \left(\frac{f_i(u) + \langle \omega_i, u \rangle}{g_i(u) - \langle \tilde{\omega}_i, u \rangle} \right).$$

Using $\lambda_i > 0, i = 1, \ldots, p$, $\sum_{i=1}^{p} \lambda_i = 1$, $\mu_j \geq 0$, $j = 1, \ldots, m$, we get

$$\sum_{i=1}^{p} \lambda_i \left(\frac{f_i(x) + \langle \omega_i, x \rangle}{g_i(x) - \langle \tilde{\omega}_i, x \rangle} \right) < \sum_{i=1}^{p} \lambda_i \left(\frac{f_i(u) + \langle \omega_i, u \rangle}{g_i(u) - \langle \tilde{\omega}_i, u \rangle} \right),$$

that is

$$\sum_{i=1}^{p} \lambda_i \left(\frac{f_i(x) + \langle \omega_i, x \rangle}{g_i(x) - \langle \tilde{\omega}_i, x \rangle} \right) - \sum_{i=1}^{p} \lambda_i \left(\frac{f_i(u) + \langle \omega_i, u \rangle}{g_i(u) - \langle \tilde{\omega}_i, u \rangle} \right) < 0 \tag{4.3}$$

and

$$-\sum_{j=1}^{m} \mu_j h_j(u) = 0. \tag{4.4}$$

By (V, ρ, σ)-type I invexity, we have

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i \nabla \left(\frac{f_i(u) + \langle \omega_i, u \rangle}{g_i(u) - \langle \tilde{\omega}_i, u \rangle} \right) \eta(x, u) + \sum_{i=1}^{p} \lambda_i \frac{\rho_i \|\bar{\theta}_i(x, u)\|^2}{\bar{\alpha}_i(x, u)} \\ &\leq \sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + \langle \omega_i, x \rangle}{g_i(x) - \langle \tilde{\omega}_i, x \rangle} \right) - \sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x, u)} \left(\frac{f_i(u) + \langle \omega_i, u \rangle}{g_i(u) - \langle \tilde{\omega}_i, u \rangle} \right) < 0 \end{aligned}$$

and

$$\sum_{j=1}^{m} \mu_j \nabla h_j(u) \eta(x, u) + \sum_{j=1}^{m} \mu_j \frac{\sigma_j \|\theta_j(x, u)\|^2}{\beta_j(x, u)} \leq -\sum_{j=1}^{m} \frac{\mu_j}{\beta_j(x, u)} \nabla h_j(u) = 0.$$

That is,

$$\sum_{i=1}^{p} \lambda_i \nabla \left(\frac{f_i(u) + \langle \omega_i, u \rangle}{g_i(u) - \langle \tilde{\omega}_i, u \rangle} \right) \eta(x, u) + \sum_{i=1}^{p} \lambda_i \frac{\rho_i \|\bar{\theta}_i(x, u)\|^2}{\bar{\alpha}_i(x, u)} < 0$$

and

$$\sum_{j=1}^{m} \mu_j \nabla h_j(u) \eta(x, u) + \sum_{j=1}^{m} \mu_j \frac{\sigma_j \|\theta_j(x, u)\|^2}{\beta_j(x, u)} \leq 0.$$

By adding the above inequalities, we get

$$\left[\sum_{i=1}^{p} \lambda_i \nabla \left(\frac{f_i(u) + \langle \omega_i, u\rangle}{g_i(u) - \langle \tilde{\omega}_i, u\rangle}\right) + \sum_{j=1}^{m} \mu_j \nabla h_j(u)\right] \eta(x,u) < 0,$$

which contradicts the dual constraints (4.1). Hence (4.2) cannot hold. □

Theorem 4.2. (Strong duality) *Let $\bar{x}$ be a weakly efficient solution for* (GFP). *Then there exist $\bar{\lambda} \in \mathbb{R}^p, \bar{\mu} \in \mathbb{R}^m$ and $\bar{\omega} \in C$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\omega}, \bar{\tilde{\omega}})$ is feasible for* (DGFP). *Moreover, if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\omega}, \bar{\tilde{\omega}})$ is a weakly efficient solution for* (DGFP).

Proof. Take $\bar{x}$ a weakly efficient solution for (GFP) and suppose that $0 \notin \mathrm{co}\{\nabla h_j(\bar{x}) \,|\, j = 1, \ldots, m\}$. Then there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$ and $\bar{\omega}_i \in C_i$, $\bar{\tilde{\omega}} \in D_i$, $i = 1, \ldots, p$ such that

$$\sum_{i=1}^{p} \bar{\lambda}_i \nabla \left(\frac{f_i(\bar{x}) + \langle \bar{\omega}_i, \bar{x}\rangle}{g_i(\bar{x}) - \langle \bar{\tilde{\omega}}_i, \bar{x}\rangle}\right) + \sum_{j=1}^{m} \bar{\mu}_j \nabla h_j(\bar{x}) = 0,$$

$$\langle \bar{\omega}_i, \bar{x}\rangle = s(\bar{x}|C_i), \ \langle \bar{\tilde{\omega}}_i, \bar{x}\rangle = s(\bar{x}|D_i),$$

$$\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = 0,$$

$$\bar{\lambda}_i > 0, i = 1, \ldots, p, \sum_{i=1}^{p} \bar{\lambda}_i = 1.$$

Thus, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\omega}, \bar{\tilde{\omega}})$ is a feasible solution for (DGFP). On the other hand, by weak duality (Theorem 4.1),

$$\max\left\{\frac{f_i(\bar{x}) + s(\bar{x}|C_i)}{g_i(\bar{x}) - s(\bar{x}|D_i)} \,|\, i = 1, \ldots, p\right\} \geq \max\left\{\frac{f_i(u) + s(u|C_i)}{g_i(u) - s(u|D_i)} \,|\, i = 1, \ldots, p\right\},$$

for any feasible solution $(x, \lambda, \mu, \omega, \tilde{\omega})$ of (DGFP). Hence $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\omega}, \bar{\tilde{\omega}})$ is a weakly efficient solution for (DGFP).

□

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