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The one-dimensional supercritical Shallow Water Equations with topography

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Abstract - The aim of this article is to present suitable boundary conditions for a flow described by the one-dimensional Shallow Water Equations when the flow is supercritical in the sense described in the text below, initial condition being suitably prescribed. Existence and uniqueness of such solutions are derived on a certain interval of time during which the flow remains supercritical and the height of the flow never vanishes.

Key words and phrases : Shallow Water Equations; transparent boundary conditions; strictly dissipative boundary conditions

Mathematics Subject Classification (2000) **:** 35L60, 35L65, 35Q35

1. Introduction

In an earlier article [12], two of the authors have addressed the question of existence and uniqueness of solutions for the inviscid Shallow Water Equations on an interval in space dimension one. In [12] the flows considered were subcritical while here they are supercritical; furthermore a bottom topography is added which can lead in the supercritical case to substantial variations of the height [8]. Another difference with our earlier work is the choice of the boundary conditions which are different in the subcritical and supercritical cases. Note that the boundary conditions are based in both cases on the Uniform Kreiss-Lopatinskii condition (UKL). See [7,9], [1], and the explanations below.

We have several motivations for studying the initial and boundary value problem associated with the inviscid Shallow Water Equations. The first major motivation is related to the problem of Limited Area Models (LAMs) in geophysical fluid mechanics. Indeed such models are commonly used in geophysical fluid mechanics, by themselves or associated with (coupled to) other models, for research or commercial purposes. Such applications include weather prediction, propagation of pollution, propagation of river effluents in an estuary, propagation of run-off water in coastal areas, etc. A major difficulty with LAMs, as emphasized in the tutorial [18], is that the chosen domain of calculation has no physical meaning and therefore no physical laws are available to determine appropriate boundary conditions; besides [13], see also [2] in which numerical simulations show the effect of boundary conditions.

The problem of the boundary conditions for LAMs was already known in the early times of weather prediction, by e.g. J. von Neumann and J. Charney. The remedy that they proposed and which is still used to some extent, is to add some viscous layers near the boundary; such approach is also extensively used in acoustics and electromagnetism, see e.g. [5,6,10,11]. Such approximation were considered as acceptable as long as the errors that they introduce are comparable to the precision of the model. It is however believed that such boundary conditions will introduce unacceptable errors when high resolution models will be used with the expected advances in computing power.

In the absence of physical laws to derive the boundary conditions which are suitable for LAMs, we will recourse to other criteria for modeling. Namely we will want the proposed boundary conditions to produce a mathematically well-posed problem and, on the physical and computational sides, we will want the boundary conditions to be transparent, that is, the boundary conditions would let the waves move freely through the boundary without producing undesirable reflexions.

Our preoccupation here relates to the mathematical well-posedness. However the boundary conditions that we propose leading to a (mildly) dissipative system, we believe that they are also transparent, a conjecture which is confirmed by the numerical experiments, see [8], [2]. We consider flows on $0 < x < L$, which are supercritical (see the details below), and in which the height and the velocity in the x direction remains always positive. As in the general theories developed in [1], the flows that we consider remain close to a stationary solution and are defined locally in time. Furthermore, an additional difficulty appears here as in [12]. Indeed, we need to restrict ourselves to either sub- or supercritical flows, the subcritical case was considered in [12], the supercritical case is considered in this article, which of course produces (necessitates) a different type of boundary conditions. Hence during the construction of the solution we require the system to remain supercritical and hyperbolic through the condition (2.8) below.

This article is organized as follows. In section 2 we describe the setting of the problem and state the main existence and uniqueness results concerning the linearized and the fully nonlinear equations. In sections 3 and 4 these theorems are proved through a classical iteration procedure.

2. Setting of the problem

In this article, we consider the following inviscid shallow water system with Coriolis force and bottom topography:

$$\begin{cases} u_t + uu_x + gh_x - fv = -gB_x, \\ v_t + uv_x + fu = 0, \\ h_t + uh_x + hu_x = 0; \end{cases} \tag{2.1}$$

here $x \in \Omega = (0, L), t \in (0, T)$, u and v are the two horizontal components of the velocity, $B = B(x)$ is the bottom function of the topography, h is the height of the water, g is the gravitational acceleration and f is the Coriolis parameter. Equations (2.1) involve the two horizontal components of the velocity, but all quantities only depend on the x variable. The first and second equations of (2.1) are derived from the conservation of momentum, and the third one expresses the conservation of mass.

2.1. Stationary solution

As indicated in the introduction we want to study system (2.1) near a stationary solution (as in [1]), and we start by constructing this stationary solution $(u, v, h) = (u_s, v_s, h_s)$. These functions are independent of time and satisfy

$$\begin{cases} uu_x + gh_x - fv = -gB_x, \\ uv_x + fu = 0, \\ (uh)_x = 0. \end{cases} \tag{2.2}$$

We infer from (2.2) that

$$\begin{cases} uh = \kappa_2, \\ v = -fx + \kappa_1, \\ u^2 + 2gh = -f^2x^2 - 2gB + 2f\kappa_1 x + \kappa_0, \end{cases}$$

where $\kappa_0, \kappa_1, \kappa_2$ are constants. We first choose $\kappa_1 = 0, \kappa_2 = 1$, and then we have $h = u^{-1}, v = -fx$ and

$$u^2 + \frac{2g}{u} = -2gB - f^2x^2 + \kappa_0. \tag{2.3}$$

Notice that $-2gB - f^2x^2$ is bounded in Ω, so we can choose κ_0 sufficiently large so that one solution of (2.3) is greater than g and bounded from above. We choose such a solution u and then $h = u^{-1} \leq g^{-1}$ which is also bounded away from zero, and furthermore $u^2 - gh \geq g^2 - 1$. All these calculations mean that we can choose our stationary solution u_s, v_s, h_s satisfying a strong form of the supercritical condition $u^2 - gh > 0$ and furthermore $u > 0, h > 0$; namely we choose u_s, v_s, h_s such that

$$\begin{cases} u_s^2 - gh_s \geq 3c_0^2, \\ u_s \geq 3a_0, \\ 3\underline{h}_0 \leq h_s \leq \overline{h}_0, \end{cases} \tag{2.4}$$

where $c_0, a_0, \underline{h}_0, \overline{h}_0$ are given, positive constants.

We set $u = u_s + \tilde{u}, v = v_s + \tilde{v}, h = h_s + \tilde{h}$, and substitute these values into (2.1); we obtain a new system for $\tilde{u}, \tilde{v}, \tilde{h}$, and dropping the tildes, our new system reads:

$$\begin{cases} u_t + (u + u_s)(u_x + u_{s,x}) + g(h_x + h_{s,x}) - f(v + v_s) = -gB_x, \\ v_t + (u + u_s)(v_x + v_{s,x}) + f(u + u_s) = 0, \\ h_t + (u + u_s)(h_x + h_{s,x}) + (h + h_s)(u_x + u_{s,x}) = 0. \end{cases} \tag{2.5}$$

We supplement system (2.5) with the following initial and boundary conditions:

$$\text{I.C.} \begin{cases} u(0, x) = u_0(x), \\ v(0, x) = v_0(x), \\ h(0, x) = h_0(x), \end{cases} \qquad \text{B.C.} \begin{cases} u(t, 0) = g_u(t), \\ v(t, 0) = g_v(t), \\ h(t, 0) = g_h(t). \end{cases}$$

The precise hypotheses on $u_0, v_0, h_0, g_u, g_v, g_h$ will be given below. Note that the boundary conditions are imposed at $x = 0$, unlike in [12] where some boundary conditions are imposed at $x = 0$ and some at $x = L$. This choice of the boundary conditions will be justified by our analysis below and the existence and uniqueness theorems (Theorem 2.1, Theorem 2.2). We choose initial conditions which are close to the stationary solution and satisfy relations similar to (2.4), that is:

$$\begin{cases} (u_0 + u_s)^2 - g(h_0 + h_s) \geq 2c_0^2, \\ u_0 + u_s \geq 2a_0, \\ 2\underline{h}_0 \leq h_0 + h_s \leq 2\overline{h}_0. \end{cases} \tag{2.6}$$

The disadvantage of this new formulation (2.5) is that the initial condition is non-zero. To overcome this difficulty, we use (u_a, v_a, h_a) an approximate lifting of the initial condition (u_0, v_0, h_0), which is given by Lemma 2.1 below, and we choose δ sufficiently small so that if

$$\begin{cases} |u_a - u_0|, |v_a - v_0|, |h_a - h_0| < \delta, \\ |u|, |v|, |h| < \delta, \end{cases} \tag{2.7}$$

then:

$$\begin{cases} (u_a + u + u_s)^2 - g(h_a + h + h_s) \geq c_0^2, \\ u_a + u + u_s \geq a_0, \\ \underline{h}_0 \leq h_a + h + h_s \leq 3\overline{h}_0. \end{cases} \tag{2.8}$$

As we will see below, the estimates (2.8) will guarantee that we remain in the supercritical case. Let us first denote:

$$U = \begin{pmatrix} u \\ v \\ h \end{pmatrix}, \quad \phi(U,x) = \begin{pmatrix} fv - gB_x \\ -fu \\ 0 \end{pmatrix}, \quad G(t) = \begin{pmatrix} g_u(t) \\ g_v(t) \\ g_h(t) \end{pmatrix},$$

$$L_U V = V_t + A(U)V_x, \quad A(U) = \begin{pmatrix} u & 0 & g \\ 0 & u & 0 \\ h & 0 & u \end{pmatrix}.$$

Using our short notations, system (2.5) is now equivalent to

$$\begin{cases} L_{U+U_s}(U+U_s) = \phi(U+U_s,x), \\ U|_{t=0} = U_0, \\ U|_{x=0} = G(t), \end{cases} \tag{2.9}$$

and we have

$$L_{U_s}(U_s) = \phi(U_s,x). \tag{2.10}$$

In order to justify the choice of the boundary conditions, let us compute the eigenvalues of matrix $A(U)$:

$$\lambda_1 = u - \sqrt{gh}, \ \lambda_2 = u, \quad \lambda_3 = u + \sqrt{gh}. \tag{2.11}$$

Since we placed ourselves in the supercritical case, all the three eigenvalues are positive and so we need to impose three boundary conditions at $x = 0$ and no boundary condition at $x = L$. This is the reason for which we prescribe the values of u, v and h at $x = 0$.

2.2. Conditions on the data

In order to be able to solve system (2.9) we need some technical conditions. First we require that $U = 0$ is a stationary solution of (2.9) with $G(0)$ as boundary data at $x = 0$, that is, in view of (2.10), we require:

$$G(0) = 0. \tag{2.12}$$

The second condition is that the initial and boundary data should satisfy some compatibility conditions. Let us rewrite the first equation of (2.9) as

$$U_t + A(U+U_s)U_x = \phi(U+U_s,x) - A(U+U_s)U_{s,x} =: H(U+U_s),$$

which is equivalent to

$$U_t = H(U+U_s) - A(U+U_s)U_x. \tag{2.13}$$

Now differentiating (2.13) in time, we see by induction that the $V_i := \partial_t^i U$ satisfy:

$$\begin{cases} V_0 = U, \\ V_1 = \partial_t U = H(U+U_s) - A(U+U_s)U_x, \\ V_{i+1} = \partial_t^{i+1} U = \sum_{k=1}^{i} \sum_{j_1+\cdots+j_k=i} c_{j_1,\cdots,j_k} (d^k H(U+U_s)) \cdot (V_{j_1}, \cdots, V_{j_k}) \\ -\sum_{l=1}^{i} \binom{i}{l} \sum_{k=1}^{l} \sum_{j_1+\cdots+j_k=l} c_{j_1,\cdots,j_k} (d^k A(U+U_s)) \cdot (V_{j_1}, \cdots, V_{j_k}) V_{i-l,x} \\ \qquad -A(U+U_s)V_{i,x}, \\ \text{for all } i \in \{1, \cdots, m-2\}; \end{cases} \tag{2.14}$$

where the coefficients $c_{j_1,\cdots,j_k}$ are derived from the Faà di Bruno's formula, see [3, 4]. Then the compatibility conditions that we require read:

$$\partial_t^p G(t)|_{t=0} = V_p(t=0, x=0), \forall p \in \{0, \cdots, m-1\}. \tag{2.15}$$

They express the classical compatibility conditions which are necessary for the solutions U of (2.9) to be $\mathcal{C}^{m-1}$ near $t = 0$; see e.g. [14, 15, 17].

2.3. Approximate Solution

Approximate solutions and related estimates are obtained by the following lemmas (for details see Chapter 11 in [1]):

Lemma 2.1. *Given $U_0 = (u_0, v_0, h_0) \in H^{m+1/2}(0, L)$, and $B \in H^{m+1}(0, L)$, there exist $T_0 > 0$ and $U_a \in H^{m+1}((0, L) \times \mathbb{R})$, vanishing for $|t| \geq 2T_0$ and such that $U_a|_{t=0} = U_0$,*

$$|U_a - U_0| < \delta, \quad \textit{for any } (x,t) \in (0, L) \times [-T_0, T_0]. \tag{2.16}$$

If we let $F^0 = -L_{U_a+U_s}(U_a + U_s) + \phi(U_a + U_s, x)$, then $F^0 \in H^m([0, L] \times \mathbb{R})$, and

$$\partial_t^p F^0 = 0 \textit{ at } t = 0, \textit{ for any } p \in \{1, 2, \cdots, m-1\}. \tag{2.17}$$

This lemma provides a lifting of the initial data U_0 by a function U_a which yields the flatness properties (2.17).

In what follows we shall denote by I_T the time interval $(-\infty, T)$, and Ω_T is $\Omega \times I_T$. More generally, we introduce the notation

$$F^U = \phi(U_a + U + U_s, x) - L_{U_a+U+U_s}(U_a + U_s).$$

Lemma 2.2. *We let $T \in (0, T_0]$, and $U \in H^m(\Omega_T)$, where $m \geq 3$, and such that*

$$\|U\|_{H^m(\Omega_T)} \leq M \in (0, \frac{\delta}{\nu_m}], \;\; U|_{t\leq 0} = 0,$$

where ν_m denotes the norm of the Sobolev embedding

$$H^m(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2).$$

Then we have

$$\partial_t^p (F^U)|_{t=0} = 0, \quad \textit{for any } p \in \{0, \cdots, m-1\},$$

and

$$\|F^U\|_{H^m(\Omega_T)} \leq C(M).$$

Remark 2.1. In Lemma 2.2, the $H^m(\Omega_T)$-norm of U is less than M; then by the Sobolev embedding, the $L^\infty(\Omega_T)$ norm of U is less than δ, and together with (2.16) in Lemma 2.1, we see that this U will stay in our admissible set, see (2.7) and (2.8).

Remark 2.2. In Lemma 2.2, all time derivatives of F^U up to order $m-1$ vanish at time $t = 0$, and then by the Cauchy-Schwarz (Poincaré) inequality, we obtain

$$\|\partial_t^p \partial_x^\beta F^U\|_{L^2(\Omega_T)} \leq T \|\partial_t^{p+1} \partial_x^\beta F^U\|_{L^2(\Omega_T)},$$

for all $p \in \{0, \cdots, m-1\}$ and all β less than or equal to $m - 1 - p$. This shows that

$$\|F^U\|_{H^{m-1}(\Omega_T)} \leq T \|F^U\|_{H^m(\Omega_T)},$$

and similar estimates also hold for U.

We recall that we set $U = U_s + \widetilde{U}$ and that we dropped the tilde in the above. Now we let $\widetilde{U} = U_a + \overline{U}$, so that $U = U_s + U_a + \overline{U}$. We then substitute it into system (2.9), drop the bar and use the notation F^U; therefore the new system becomes:

$$\begin{cases} L_{U_a+U+U_s} U = \phi(U_a + U + U_s, x) - L_{U_a+U+U_s}(U_a + U_s) = F^U, \\ U|_{t=0} = 0, \\ U|_{x=0} = -U_a + G(t) =: G^0(t). \end{cases} \tag{2.18}$$

Applying ∂_t^p to the first equation (2.18), we find

$$\partial_t^{p+1} U + \partial_t^p (A(U_a + U + U_s) U_x) = \partial_t^p F^U,$$

which is equivalent to

$$\partial_t^{p+1} U = \partial_t^p F^U - \sum_{0 \le \alpha \le p} \partial_t^{p-\alpha}(A(U_a + U + U_s))\partial_t^\alpha U_x.$$

By Lemma 2.2, we have

$$\partial_t^p F^U|_{t=0} = 0, \text{ for any } p \in \{0, \cdots, m-1\},$$

and we also have $U|_{t=0} = 0$, so by induction, we see that

$$\partial_t^p U|_{t=0} = 0, \text{ for any } p \in \{0, \cdots, m\}.$$

Hence we can extend U by zero for $t < 0$ in $H^m(\Omega_T)$, and we see that the first equation (2.18) is valid for $(x,t) \in \Omega_T$, the second equation (2.18) is valid for $t \le 0$, and the third equation (2.18) is valid for $t \le T$.

Based on these observations, we now introduce our iterative scheme defined as follows:

$$\begin{cases} L_{U_a+U^k+U_s} U^{k+1} = U_t^{k+1} + A(U_a + U^k + U_s)U_x^{k+1} = F^{U^k}, (x,t) \in \Omega_T, \\ U^{k+1}|_{t\le 0} = 0, \\ U^{k+1}|_{x=0} = -U_a + G(t) = G^0(t), t \le T. \end{cases} \tag{2.19}$$

We initiate our iteration scheme by setting $U^0 = 0$, and then construct the U^k by induction. In Section 3 we prove the existence of the U^k and some uniform bounds on the sequence $\{U^k\}$ which is given by Theorem 2.1 below. Then in Section 4 we prove the strong convergence of the sequence $\{U^k\}$ which finally gives Theorem 2.2 below.

Theorem 2.1 (For the linear system (2.19)) *We assume that one has $U^k \in H^m(\Omega_T)$, $m \ge 3, U^k|_{t\le 0} = 0$, and satisfies*

$$\|U^k\|_{H^m(\Omega_T)} \le M, \text{ with } M \in (0, \frac{\delta}{\nu_m}]. \tag{2.20}$$

We also assume that $G(\cdot)$ (see (2.9)) belongs to $H^m(I_T)$, so that the boundary data $G^0(\cdot)$ (see (2.18)) belongs to $H^m(I_T)$ since $U_a \in H^{m+1}(\Omega_T)$. We also assume that the forcing term F^{U^k} satisfies the properties from Lemma 2.2. Then the initial boundary value problem (2.19) has a unique solution U^{k+1} such that

$$\|U^{k+1}\|_{H^m(\Omega_T)} \le M. \tag{2.21}$$

Furthermore by the Sobolev embedding, we have

$$\|U^{k+1}\|_{L^\infty(\Omega_T)} \le \delta. \tag{2.22}$$

Theorem 2.2 (For the nonlinear system (2.5)) *Let there be given* $U_0 = (u_0, v_0, h_0)$ *in* $H^{m+\frac{1}{2}}(\Omega)$ *satisfying the condition* (2.6)*,* B *in* $H^{m+1}(\Omega)$ *and* $G(\cdot) = (g_u(\cdot),\, g_v(\cdot), g_h(\cdot))$ *in* $H^m(I_T)$ *such that* $G(0) = 0$ *and satisfying the compatibility conditions* (2.15)*. Then there exists* $T_* > 0$ *depending on the initial and boundary data and also on the stationary solutions* U_s *such that the system* (2.5) *admits a unique solution* $\widetilde{U} = (\tilde{u}, \tilde{v}, \tilde{h})$ *satisfying*

$$\widetilde{U} \in H^m((0, L) \times (0, T_*)). \tag{2.23}$$

3. High norm boundedness of U^k

In this section we will prove Theorem 2.1. We will first derive a priori L^2-estimates for the solution of the linear system (2.19) and then extend the L^2-estimates to H^m-estimates and finally by choosing T small enough, we will obtain the uniform bound on U^k.

We want the solutions of (2.19) to stay always in the admissible set, i.e. we want them to satisfy:

$$\begin{cases} (u_a + u^k + u_s)^2 - g(h_a + h^k + h_s) \geq c_0^2, \\ u_a + u^k + u_s \geq a_0, \\ \underline{h}_0 \leq h_a + h^k + h_s \leq 3\overline{h}_0, \end{cases} \tag{3.1}$$

which will be achieved by controlling the L^∞-norm of U^k (See (2.8) and Remark 2.1).

New notation: we write $\widehat{U} = \widehat{U}^k = U_a + U^k + U_s$, then $A(\widehat{U}) = A(U_a + U^k + U_s)$. Observing that system (2.19) admits a symmetrizer $S_0 = \mathrm{diag}(h_a + h^k + h_s, 1, g) = \mathrm{diag}(\widehat{h}, 1, g)$, we have

$$S_0 A(\widehat{U}) = \begin{pmatrix} \hat{u}\hat{h} & 0 & g\hat{h} \\ 0 & \hat{u} & 0 \\ g\hat{h} & 0 & g\hat{u} \end{pmatrix}; \tag{3.2}$$

we denote the eigenvalues of $S_0 A$ by $\lambda_1, \lambda_2, \lambda_3$, which are the roots of the equation

$$\det(\lambda I_3 - S_0 A) = (\lambda - \hat{u})(\lambda^2 - (\hat{u}\hat{h} + g\hat{u})\lambda + g\hat{h}(\hat{u}^2 - g\hat{h})) = 0.$$

Let us say λ_3 is $\hat{u}$, and then λ_1, λ_2 satisfy $\lambda_1 + \lambda_2 = \hat{u}\hat{h} + g\hat{u}, \lambda_1\lambda_2 = g\hat{h}(\hat{u}^2 - g\hat{h})$. If U^k satisfies (3.1), then $\lambda_3 > 0$, and $\lambda_1 + \lambda_2 > 0, \lambda_1\lambda_2 > 0$, which implies $\lambda_1 > 0, \lambda_2 > 0$. Therefore $S_0 A$ is symmetric positive definite if U^k stays in the admissible set (that is it satisfies (3.1)).

Now we multiply (2.19) by S_0 and take the scalar product with U^{k+1}.

Using integration by parts, we first obtain

$$\begin{aligned}\langle S_0U_t^{k+1}, U^{k+1}\rangle_{L^2(\Omega)} &= \frac{1}{2}\frac{d}{dt}\int_\Omega \hat{h}(u^{k+1})^2 + (v^{k+1})^2 + g(h^{k+1})^2 dx \\ &\quad - \frac{1}{2}\int_\Omega (h_{a,t} + h_t^k)(u^{k+1})^2 dx, \\ \langle S_0AU_x^{k+1}, U^{k+1}\rangle_{L^2(\Omega)} &= \frac{1}{2}\langle S_0AU_x^{k+1}, U^{k+1}\rangle\Big|_{x=0}^{x=L} \\ &\quad - \frac{1}{2}\langle (S_0A)_x U^{k+1}, U^{k+1}\rangle_{L^2(\Omega)}.\end{aligned}$$

Finally, we find the following equation:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \hat{h}(u^{k+1})^2 + (v^{k+1})^2 + g(h^{k+1})^2 dx + \frac{1}{2}\langle S_0AU_x^{k+1}, U^{k+1}\rangle\big|_{x=L} \\ &\quad = \frac{1}{2}\int_\Omega (h_{a,t} + h_t^k)(u^{k+1})^2 dx + \frac{1}{2}\langle (S_0A)_x U^{k+1}, U^{k+1}\rangle_{L^2(\Omega)} \\ &\qquad + \frac{1}{2}\langle S_0AU_x^{k+1}, U^{k+1}\rangle\big|_{x=0} + \langle S_0F^{U^k}, U^{k+1}\rangle_{L^2(\Omega)}.\end{aligned} \tag{3.3}$$

Using that S_0A is positive definite, we neglect the boundary term at $x = L$ in the left-hand side of (3.3). We obtain the following inequality:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \hat{h}(u^{k+1})^2 + (v^{k+1})^2 + g(h^{k+1})^2 dx \\ &\quad \le \frac{1}{2}\int_\Omega |h_{a,t} + h_t^k|(u^{k+1})^2 dx + \frac{1}{2}|\langle (S_0A)_x U^{k+1}, U^{k+1}\rangle_{L^2(\Omega)}| \\ &\qquad + \frac{1}{2}|\langle S_0AU_x^{k+1}, U^{k+1}\rangle\big|_{x=0}| + |\langle S_0F^{U^k}, U^{k+1}\rangle_{L^2(\Omega)}|.\end{aligned} \tag{3.4}$$

Let us first denote

$$I_0(t) = \int_\Omega \hat{h}(u^{k+1})^2 + (v^{k+1})^2 + g(h^{k+1})^2 dx, \tag{3.5}$$

and using (3.1) for $\hat{h} = h_a + h^k + h_s$, we easily see that

$$I_0(t) \ge \min(\underline{h}_0, 1, g)\|U^{k+1}\|^2_{L^2(\Omega)}. \tag{3.6}$$

We now estimate the four terms J_1, J_2, J_3, J_4 in the right-hand side of (3.4).

Using (3.1) again and $H^2(\Omega_T) \subset L^\infty(\Omega_T)$, we find

$$\begin{aligned}
J_1 &\leq \frac{1}{2}\|h_{a,t} + h_t^k\|_{L^\infty(\Omega_T)} \int_\Omega \frac{1}{h_a + h^k + h_s}(h_a + h^k + h_s)(u^{k+1})^2 dx \\
&\leq \frac{1}{2\underline{h}_0}\|h_a + h^k\|_{H^3(\Omega_T)} \int_\Omega \hat{h}(u^{k+1})^2 dx \\
&\leq C\Big(\|\hat{h}\|_{H^3(\Omega_T)}\Big) I_0(t), \\
J_2 &= \frac{1}{2}\int_\Omega (\hat{h}\hat{u})_x (u^{k+1})^2 + g\hat{u}_x (h^{k+1})^2 + \hat{u}_x (v^{k+1})^2 + 2g\hat{h}_x u^{k+1} h^{k+1} dx \\
&\leq \frac{1}{2}\Big[\|(\hat{h}\hat{u})_x\|_{L^\infty(\Omega_T)} \frac{1}{\underline{h}_0}\int_\Omega \hat{h}(u^{k+1})^2 dx \\
&\quad + \|\hat{u}_x\|_{L^\infty(\Omega_T)} \int_\Omega g(h^{k+1})^2 + (v^{k+1})^2 dx \\
&\quad + \|g\hat{h}_x\|_{L^\infty(\Omega_T)} \big(\frac{1}{\underline{h}_0}\int_\Omega \hat{h}(u^{k+1})^2 dx + \frac{1}{g}\int_\Omega g(h^{k+1})^2 dx\big)\Big] \\
&\leq C(\|\hat{h}_x\|_{L^\infty(\Omega_T)}, \|\hat{u}_x\|_{L^\infty(\Omega_T)}) I_0(t) \\
&\leq C(\|\hat{h}\|_{H^3(\Omega_T)}, \|\hat{u}\|_{H^3(\Omega_T)}) I_0(t).
\end{aligned}$$

Using (3.2) and the boundary data at $x = 0$, we find

$$\begin{aligned}
J_3 &\leq \frac{1}{2}\|\hat{h}\hat{u}\|_{L^\infty(I_T)} |g_u^0(t)|^2 + \frac{1}{2}\|g\hat{u}\|_{L^\infty(I_T)} |g_h^0(t)|^2 + \frac{1}{2}\|\hat{u}\|_{L^\infty(I_T)} |g_v^0(t)|^2 \\
&\quad + \|g\hat{h}\|_{L^\infty(I_T)} |g_u^0(t)||g_h^0(t)| \\
&\leq C(\|\hat{h}\|_{L^\infty(\Omega_T)}, \|\hat{u}\|_{L^\infty(\Omega_T)}) |G^0(t)|^2 \\
&\leq C(\|\hat{h}\|_{H^3(\Omega_T)}, \|\hat{u}\|_{H^3}) |G^0(t)|^2.
\end{aligned}$$

By Cauchy-Schwarz inequality and using that $S_0 = \mathrm{diag}(\hat{h}, 1, g)$, we find

$$\begin{aligned}
J_4 = |\langle S_0 F^{U^k}, U^{k+1}\rangle_{L^2(\Omega)}| &\leq \frac{1}{4}\|S_0 F^{U^k}\|^2_{L^2(\Omega)} + \|U^{k+1}\|^2_{L^2(\Omega)} \\
&\leq \frac{1}{4}\max(\|\hat{h}\|^2_{L^\infty(\Omega_T)}, 1, g^2)\|F^{U^k}\|^2_{L^2(\Omega)} + \max(\frac{1}{\underline{h}_0}, 1, \frac{1}{g}) I_0(t) \\
&\leq C(\|\hat{h}\|_{H^3(\Omega_T)})\|F^{U^k}\|^2_{L^2(\Omega)} + \max(\frac{1}{\underline{h}_0}, 1, \frac{1}{g}) I_0(t).
\end{aligned}$$

Gathering all the above estimates, inequality (3.4) becomes

$$\frac{d}{dt} I_0(t) \leq C_1(\hat{u}, \hat{h}) I_0(t) + C_2(\hat{h})\|F^{U^k}(t)\|^2_{L^2(\Omega)} + C_3(\hat{u}, \hat{h})|G^0(t)|^2, \tag{3.7}$$

where the $C_i(\hat{u}, \hat{h})(i = 1, 2, 3)$ only depend on $\|\hat{u}\|_{H^3(\Omega_T)}, \|\hat{h}\|_{H^3(\Omega_T)}$, and in an increasing way. Observe that $C_i(\hat{u}, \hat{h}) \leq C_i(\|\hat{u}\|_{H^m(\Omega_T)}, \|\hat{h}\|_{H^m(\Omega_T)}) \leq C_i(M)$, with $m \geq 3$ by assumption.

By Gronwall's lemma, noticing that $I_0(0) = 0$, which comes from the initial condition, we obtain

$$I_0(t) \leq \int_0^t e^{C_1(M)(t-\tau)}(C_2(M)\|F^{U^k}(\tau)\|^2_{L^2(\Omega)} + C_3(M)|G^0(\tau)|^2)d\tau. \quad (3.8)$$

Let us set

$$r_1 = C_1(M);\ \ r_2 = \frac{\max(C_2(M), C_3(M))}{\min(\underline{h}_0, 1, g)}.$$

Therefore r_1, r_2 only depend on M, which is a bound on the $H^m(\Omega_T)$-norm of U^k ($m \geq 3$), and they are independent of k, so that r_1, r_2 will be the same at every iteration.

Using (3.6), inequality (3.8) becomes

$$\|U^{k+1}(t)\|^2_{L^2(\Omega)} \leq r_2 \int_0^t e^{r_1(t-\tau)}(\|F^{U^k}(\tau)\|^2_{L^2(\Omega)} + |G^0(\tau)|^2)d\tau; \quad (3.9)$$

integrating over $[0, T]$ and using Fubini theorem, we find

$$\|U^{k+1}\|^2_{L^2(\Omega_T)} \leq \frac{e^{r_1 T} - 1}{r_1} r_2(\|F^{U^k}\|^2_{L^2(\Omega_T)} + \|G^0\|^2_{L^2(I_T)}). \quad (3.10)$$

Writing $\epsilon(T) = (e^{r_1 T} - 1)r_2/r_1$, we see that $\lim_{T\to 0} \epsilon(T) = 0$, and (3.10) gives

$$\|U^{k+1}\|^2_{L^2(\Omega_T)} \leq \epsilon(T)(\|F^{U^k}\|^2_{L^2(\Omega_T)} + \|G^0\|^2_{L^2(I_T)}). \quad (3.11)$$

3.1. Tangential Derivatives

We now want to estimate the tangential derivatives of U^k; in our case, the tangential derivatives are the derivatives with respect to time. We set $U^{k+1}_\alpha = \partial_t^\alpha U^{k+1}$ with $0 \leq \alpha \leq m$, and deduce from equation (2.19) that U^{k+1}_α satisfies the following equations:

$$\begin{cases} (U^{k+1}_\alpha)_t + A(\widehat{U})(U^{k+1}_\alpha)_x = \partial_t^\alpha F^{U^k} - [\partial_t^\alpha, A(\widehat{U})]U^{k+1}_x =: \widetilde{F_\alpha}, \\ U^{k+1}_\alpha|_{t\leq 0} = 0, \\ U^{k+1}_\alpha|_{x=0} = \partial_t^\alpha G^0(t), \end{cases} \quad (3.12)$$

where $[\cdot,\cdot]$ denotes the commutator (i.e. $[P, Q] = PQ - QP$).

Observing that equation (3.12) has the same form as (2.19), therefore proceeding exactly as for (3.11), we find:

$$\|U^{k+1}_\alpha\|^2_{L^2(\Omega_T)} \leq \epsilon(T)(\|\widetilde{F_\alpha}\|^2_{L^2(\Omega_T)} + \|\partial_t^\alpha G^0\|^2_{L^2(I_T)}). \quad (3.13)$$

In order to estimate $\widetilde{F_\alpha}$, we need the following classical lemmas (see e.g. Chapter 13 in [16]):

Lemma 3.1. *Assume that $\mathcal{U}$ is a regular open set of $\mathbb{R}^d$, where d is the dimension of the space.*

(i) Consider u and v which both belong to $L^\infty(\mathcal{U}) \cap H^s(\mathcal{U})$ with $s > 0$, then their product also belongs to $H^s(\mathcal{U})$ and there exists $C > 0$ depending only on s and $\mathcal{U}$ such that

$$\|uv\|_{H^s(\mathcal{U})} \leq C(\|u\|_{L^\infty(\mathcal{U})}\|v\|_{H^s(\mathcal{U})} + \|v\|_{L^\infty(\mathcal{U})}\|u\|_{H^s(\mathcal{U})}).$$

If $s > d/2$, then the L^∞ assumption automatically follows from the Sobolev embeddings, and we have the following estimate:

$$\|uv\|_{H^s(\mathcal{U})} \leq C\|u\|_{H^s(\mathcal{U})}\|v\|_{H^s(\mathcal{U})}.$$

(ii) If m is an integer greater than $d/2+1$ and α is a d-tuple of length $|\alpha| \in [1, m]$, there exists $C > 0$ depending only on s and $\mathcal{U}$ such that for all a in $H^m(\mathcal{U})$ and all $u \in H^{|\alpha|-1}(\mathcal{U})$, we have the following estimate:

$$\|[\partial^\alpha, a]u\|_{L^2(\mathcal{U})} \leq C\|a\|_{H^m(\mathcal{U})}\|u\|_{H^{\alpha-1}(\mathcal{U})}.$$

(iii) For all s and t with $s + t > 0$, if $u \in H^s(\mathcal{U})$ and $v \in H^t(\mathcal{U})$, then the product uv belongs to $H^r(\mathcal{U})$ for all $r \leq min(s,t)$ such that $r < s + t - d/2$. Furthermore, there exists C (depending only on r, s, t, d and $\mathcal{U}$) such that

$$\|uv\|_{H^r(\mathcal{U})} \leq C\|u\|_{H^s(\mathcal{U})}\|v\|_{H^t(\mathcal{U})}.$$

Now we are able to estimate $\widetilde{F_\alpha}$; we first claim that $A^{-1}(\widehat{U}) \in H^m(\Omega_T)$. We know that

$$A^{-1}(\widehat{U}) = \frac{1}{\hat{u}^2 - g\hat{h}} \begin{pmatrix} \hat{u} & 0 & -g \\ 0 & \frac{1}{\hat{u}} & 0 \\ -\hat{h} & 0 & \hat{u} \end{pmatrix}.$$

In our case, we have $m \geq 3, d = 2$, thus $H^m(\Omega_T) \subset L^\infty(\Omega_T)$. Notice also that $\hat{u}^2 - g\hat{h} \geq c_0^2$ and $\hat{u} > a_0$, and notice that our domain is bounded, so that $(\hat{u}^2 - g\hat{h})^{-1}$ and $\hat{u}^{-1}$ both belong to $H^m(\Omega_T)$, and then by Lemma 3.1 (i), we find that $A^{-1}(\widehat{U}) \in H^m(\Omega_T)$, and $\|A^{-1}(\widehat{U})\|_{H^m(\Omega_T)} \leq C(\|\widehat{U}\|_{H^m(\Omega_T)}) \leq C(M)$.

Now we rewrite system (2.19) as $U_x^{k+1} = A^{-1}(\widehat{U})(F^{U^k} - U_t^{k+1})$ and with the first equation (3.12), we find

$$\widetilde{F_\alpha} = \partial_t^\alpha F^{U^k} - [\partial_t^\alpha, A(\widehat{U})][A^{-1}(\widehat{U})F^{U^k}] + [\partial_t^\alpha, A(\widehat{U})][A^{-1}(\widehat{U})U_t^{k+1}]. \quad (3.14)$$

We estimate $\sum_{0\leq\alpha\leq m}\|\widetilde{F_\alpha}\|^2_{L^2(\Omega_T)}$; the three terms J_1, J_2, J_3 in the right-hand side of (3.14) give:

$$J_1 = \sum_{0\leq\alpha\leq m} \|\partial_t^\alpha F^{U^k}\|^2_{L^2(\Omega_T)} \leq \|F^{U^k}\|^2_{H^m(\Omega_T)}$$
$$\leq (\text{With Lemma 2.2}) \leq C(M).$$

Noticing that U_a, U_s are fixed, by assumption (2.20) in Theorem 2.1, the $H^m(\Omega_T)$-norm of $\widehat{U} = U_a + U^k + U_s$ is bounded by an increasing function depending only on M, and the same also holds for the $L^\infty(\Omega_T)$ and $H^m(\Omega_T)$-norms of $A(\widehat{U})$ and $A^{-1}(\widehat{U})$. Therefore we find

$$
\begin{aligned}
J_2 &= \sum_{0\le\alpha\le m} \|[\partial_t^\alpha, A(\widehat{U})][A^{-1}(\widehat{U})F^{U^k}]\|_{L^2(\Omega_T)} \\
&\le \text{(Using Lemma 3.1 (ii))} \\
&\le \sum_{0\le\alpha\le m} \|A(\widehat{U})\|^2_{H^m(\Omega_T)} \|A^{-1}(\widehat{U})F^{U^k}\|^2_{H^{\alpha-1}(\Omega_T)} \\
&\le C(M)\|A^{-1}(\widehat{U})F^{U^k}\|^2_{H^{m-1}(\Omega_T)} \\
&\le \text{(Using Lemma 3.1 (i) where } m \ge 3, d=2, s=m-1 > \frac{d}{2}) \\
&\le C(M)\|A^{-1}(\widehat{U})\|^2_{H^{m-1}(\Omega_T)} \|F^{U^k}\|^2_{H^{m-1}(\Omega_T)} \\
&\le C(M)\|F^{U^k}\|^2_{H^m(\Omega_T)} \le \text{(With Lemma 2.2)} \le C(M).
\end{aligned}
$$

The estimate for J_3 is exactly the same as for J_2:

$$
\begin{aligned}
J_3 &= \sum_{0\le\alpha\le m} \|[\partial_t^\alpha, A(\widehat{U})][A^{-1}(\widehat{U})U_t^{k+1}]\|_{L^2(\Omega_T)} \\
&\le \sum_{0\le\alpha\le m} \|A(\widehat{U})\|^2_{H^m(\Omega_T)} \|A^{-1}(\widehat{U})U_t^{k+1}\|^2_{H^{\alpha-1}(\Omega_T)} \\
&\le C(M)\|A^{-1}(\widehat{U})U_t^{k+1}\|^2_{H^{m-1}(\Omega_T)} \\
&\le C(M)\|A^{-1}(\widehat{U})\|^2_{H^{m-1}(\Omega_T)} \|U_t^{k+1}\|^2_{H^{m-1}(\Omega_T)} \\
&\le C(M)\|U^{k+1}\|^2_{H^m(\Omega_T)}.
\end{aligned}
$$

In conclusion, we have

$$
\sum_{0\le\alpha\le m} \|\widetilde{F_\alpha}\|^2_{H^m(\Omega_T)} \le C(M)(1 + \|U^{k+1}\|^2_{H^m(\Omega_T)}). \tag{3.15}
$$

Now summing the inequalities (3.13) for $\alpha = 0, \cdots, m$ and using (3.15) for $\widetilde{F_\alpha}$, we obtain

$$
\|U^{k+1}\|^2_{L^2(\Omega, H^m(I_T))} \le \epsilon(T)\Big(C(M)(1 + \|U^{k+1}\|^2_{H^m(\Omega_T)}) + \|G^0\|^2_{H^m(I_T)}\Big). \tag{3.16}
$$

3.2. Normal Derivatives

We also need to estimate the derivatives containing x, thus we need to consider all the derivatives of length $\le m$. We first consider $\partial_x \partial_t^\alpha$ with

$0 \leq \alpha \leq m-1$. From system (2.19), we deduce that:

$$\begin{aligned} \partial_x \partial_t^\alpha U^{k+1} &= \partial_t^\alpha [A^{-1}(\widehat{U})(F^{U^k} - U_t^{k+1})] = A^{-1}(\widehat{U})\partial_t^\alpha F^{U^k} \\ &+ [\partial_t^\alpha, A^{-1}(\widehat{U})]F^{U^k} - A^{-1}(\widehat{U})\partial_t^{\alpha+1}U^{k+1} - [\partial_t^\alpha, A^{-1}(\widehat{U})]U_t^{k+1}. \end{aligned} \tag{3.17}$$

We take the L^2-norm on both sides of equation (3.17), then sum for $\alpha = 0, \cdots, m-1$, and we estimate the four terms coming from the right-hand side of (3.17):

$$\begin{aligned} J_1 = & \sum_{0 \leq \alpha \leq m-1} \|A^{-1}(\widehat{U})\partial_t^\alpha F^{U^k}\|^2_{L^2(\Omega_T)} \\ \leq & \ \|A^{-1}(\widehat{U})\|^2_{L^\infty(\Omega_T)} \|F^{U^k}\|^2_{L^2(\Omega, H^{m-1}(I_T))} \\ \leq & \ C(\|\widehat{U}\|_{H^m(\Omega_T)}) \|F^{U^k}\|^2_{H^{m-1}(\Omega_T)} \\ \leq & \ (\text{By Remark 2.2}) \leq C(M)T^2 \|F^{U^k}\|^2_{H^m(\Omega_T)} \\ \leq & \ (\text{With Lemma 2.2}) \leq C(M)T^2, \end{aligned}$$

$$\begin{aligned} J_3 = & \sum_{0 \leq \alpha \leq m-1} \|A^{-1}(\widehat{U})\partial_t^{\alpha+1}U^{k+1}\|^2_{L^2(\Omega_T)} \\ \leq & \sum_{0 \leq \alpha \leq m-1} \|A^{-1}(\widehat{U})\|^2_{L^\infty(\Omega_T)} \|\partial_t^{\alpha+1}U^{k+1}\|^2_{L^2(\Omega_T)} \\ \leq & \ C(\|\widehat{U}\|_{H^m(\Omega_T)}) \|U^{k+1}\|^2_{L^2(\Omega, H^m(I_T))} \leq C(M) \|U^{k+1}\|^2_{L^2(\Omega, H^m(I_T))}, \end{aligned}$$

$$\begin{aligned} J_2 = & \sum_{0 \leq \alpha \leq m-1} \|[\partial_t^\alpha, A^{-1}(\widehat{U})]F^{U^k}\|^2_{L^2(\Omega_T)} \\ \leq & \ (\text{By Lemma 3.1 (ii)}) \leq \sum_{0 \leq \alpha \leq m-1} \|A^{-1}(\widehat{U})\|^2_{H^m(\Omega_T)} \|F^{U^k}\|^2_{H^{\alpha-1}(\Omega_T)} \\ \leq & \ C(\|\widehat{U}\|_{H^m(\Omega_T)}) \|F^{U^k}\|^2_{H^{m-2}(\Omega_T)} \leq C(M) \|F^{U^k}\|^2_{H^{m-1}(\Omega_T)} \\ \leq & \ (\text{By Remark 2.2}) \leq C(M)T^2 \|F^{U^k}\|^2_{H^m(\Omega_T)} \\ \leq & \ (\text{With Lemma 2.2}) \leq C(M)T^2, \end{aligned}$$

$$\begin{aligned} J_4 = & \sum_{0 \leq \alpha \leq m-1} \|[\partial_t^\alpha, A^{-1}(\widehat{U})]U_t^{k+1}\|^2_{L^2(\Omega_T)} \\ \leq & \ (\text{By Lemma 3.1 (ii)}) \leq \sum_{0 \leq \alpha \leq m-1} \|A^{-1}(\widehat{U})\|^2_{H^m(\Omega_T)} \|U_t^{k+1}\|^2_{H^{\alpha-1}(\Omega_T)} \\ \leq & \ C(\|\widehat{U}\|_{H^m}) \|U_t^{k+1}\|^2_{H^{m-2}(\Omega_T)} \leq C(M) \|U^{k+1}\|^2_{H^{m-1}(\Omega_T)} \\ \leq & \ (\text{By Remark 2.2}) \leq C(M)T^2 \|U^{k+1}\|^2_{H^m(\Omega_T)}. \end{aligned}$$

Using $\|\partial_x U^{k+1}\|^2_{L^2(\Omega,H^{m-1}(I_T))} = \sum_{0\le\alpha\le m-1}\|\partial_x\partial_t^\alpha U^{k+1}\|^2_{L^2(\Omega_T)}$, and using the above estimates, we arrive at

$$\begin{aligned}\|\partial_x U^{k+1}\|^2_{L^2(\Omega,H^{m-1}(I_T))} \le C(M)\big(&T^2(1+\|U^{k+1}\|^2_{H^m(\Omega_T)})\\ &+\|U^{k+1}\|^2_{L^2(\Omega,H^m(I_T))}\big).\end{aligned} \tag{3.18}$$

Using (3.16) to substitute the term $\|U^{k+1}\|^2_{L^2(\Omega,H^m(I_T))}$ in (3.18), we finally find

$$\|\partial_x U^{k+1}\|^2_{L^2(\Omega,H^{m-1}(I_T))} \le \epsilon(T)C(M)\big(1+\|U^{k+1}\|^2_{H^m(\Omega_T)}+\|G^0\|^2_{H^m(I_T)}\big), \tag{3.19}$$

where $\epsilon(T)$ is slightly different from the previous one, but enjoys the same properties – it only depends on M and T, it is independent of k, and $\lim_{T\to 0}\epsilon(T)=0$.

We also need to estimate the terms of the form $\partial_x^j\partial_t^{m-j}U^{k+1}$ for all j; we will proceed by induction. Let us first assume that we have proven the estimate:

$$\|\partial_x^j U^{k+1}\|^2_{L^2(\Omega,H^{m-j}(I_T))} \le \epsilon(T)C(M)(1+\|U^{k+1}\|^2_{H^m(\Omega_T)}+\|G^0\|^2_{H^m(I_T)}). \tag{3.20}$$

We want to prove a similar estimate for $\partial_x^{j+1}\partial_t^\alpha U^{k+1}$ with $0\le\alpha\le m-j-1$. We apply $\partial_x^j\partial_t^\alpha$ to system (2.19), and denote $\beta=(j,\alpha),\partial^\beta=\partial_x^j\partial_t^\alpha,|\beta|\le m-1$; we find

$$\begin{aligned}\partial_x^{j+1}\partial_t^\alpha U^{k+1} &= \partial^\beta[A^{-1}(\widehat U)(F^{U^k}-U_t^{k+1})] = A^{-1}(\widehat U)\partial^\beta F^{U^k}\\ &+[\partial^\beta,A^{-1}(\widehat U)]F^{U^k}-A^{-1}(\widehat U)\partial^\beta\partial_t U^{k+1}-[\partial^\beta,A^{-1}(\widehat U)]U_t^{k+1}.\end{aligned} \tag{3.21}$$

We take the L^2-norm of each term in equation (3.21), then sum for $\alpha=0,\cdots,m-j-1$, and we use the same arguments as we did for equation (3.17), to estimate the four terms coming from the right-hand side of (3.21):

$$\begin{aligned}J_1 &= \sum_{0\le\alpha\le m-j-1}\|A^{-1}(\widehat U)\partial^\beta F^{U^k}\|^2_{L^2(\Omega_T)}\\ &\le \|A^{-1}(\widehat U)\|^2_{L^\infty(\Omega_T)}\|F^{U^k}\|^2_{H^{m-1}(\Omega_T)}\\ &\le C(\|\widehat U\|_{H^m(\Omega_T)})T^2\|F^{U^k}\|^2_{H^m(\Omega_T)}\le C(M)T^2,\end{aligned}$$

$$\begin{aligned}J_3 &= \sum_{0\le\alpha\le m-j-1}\|A^{-1}(\widehat U)\partial^\beta\partial_t U^{k+1}\|^2_{L^2(\Omega_T)}\\ &\le \sum_{0\le\alpha\le m-j-1}\|A^{-1}(\widehat U)\|^2_{L^\infty(\Omega_T)}\|\partial_t^{\alpha+1}\partial_x^j U^{k+1}\|_{L^2(\Omega_T)}\\ &\le C(M)\|\partial_x^j U^{k+1}\|_{L^2(\Omega,H^{m-j}(I_T))},\end{aligned}$$

$$
\begin{aligned}
J_2 = & \sum_{0\le\alpha\le m-j-1} \|[\partial^\beta, A^{-1}(\widehat{U})]F^{U^k}\|^2_{L^2(\Omega_T)} \\
\le & \sum_{0\le\alpha\le m-j-1} \|A^{-1}(\widehat{U})\|^2_{H^m(\Omega_T)} \|F^{U^k}\|^2_{H^{|\beta|-1}(\Omega_T)} \\
\le & \ C(M)\|F^{U^k}\|^2_{H^{m-2}(\Omega_T)} \le C(M)\|F^{U^k}\|^2_{H^{m-1}(\Omega_T)} \\
\le & \ C(M)T^2\|F^{U^k}\|^2_{H^m(\Omega_T)} \le C(M)T^2,
\end{aligned}
$$

$$
\begin{aligned}
J_4 = & \sum_{0\le\alpha\le m-j-1} \|[\partial^\beta, A^{-1}(\widehat{U})]U_t^{k+1}\|^2_{L^2(\Omega_T)} \\
\le & \sum_{0\le\alpha\le m-j-1} \|A^{-1}(\widehat{U})\|^2_{H^m(\Omega_T)} \|U_t^{k+1}\|^2_{H^{|\beta|-1}(\Omega_T)} \\
\le & \ C(M)\|U_t^{k+1}\|^2_{H^{m-2}(\Omega_T)} \le C(M)\|U^{k+1}\|^2_{H^{m-1}(\Omega_T)} \\
\le & \ C(M)T^2\|U^{k+1}\|^2_{H^m(\Omega_T)}.
\end{aligned}
$$

Using the above estimates, we find

$$
\begin{aligned}
\|\partial_x^{j+1}U^{k+1}\|^2_{L^2(\Omega, H^{m-j-1}(I_T))} \le & \ C(M)(T^2(1+\|U^{k+1}\|^2_{H^m(\Omega_T)}) \\
& + \|\partial_x^j U^{k+1}\|_{L^2(\Omega, H^{m-j}(I_T))}).
\end{aligned}
$$

By the induction assumption (3.20) on the term $\|\partial_x^j U^{k+1}\|_{L^2(\Omega, H^{m-j}(I_T))}$, we obtain

$$
\|\partial_x^{j+1}U^{k+1}\|^2_{L^2(\Omega, H^{m-j-1}(I_T))} \le \epsilon(T)C(M)(1+\|U^{k+1}\|^2_{H^m(\Omega_T)} + \|G^0\|^2_{H^m(I_T)}).
$$

Thus we proved that inequality (3.20) is true for all j. Now summing these inequalities (3.20) for $j = 0, \cdots, m$, we find

$$
\|U^{k+1}\|^2_{H^m(\Omega_T)} \le \epsilon(T)C(M)(1+\|U^{k+1}\|^2_{H^m(\Omega_T)} + \|G^0\|^2_{H^m(I_T)}). \tag{3.22}
$$

Using that $\lim_{T\to 0}\epsilon(T) = 0$, we can choose T small enough to absorb the term $\|U^{k+1}\|^2_{H^m(\Omega_T)}$ on the right-hand side of (3.22) and obtain

$$
\|U^{k+1}\|_{H^m(\Omega_T)} \le M,
$$

which proves Theorem 2.1.

Remark 3.1. Let us point out that the choice of T depends only on M and on the constants $c_0, a_0, \underline{h}_0, \overline{h}_0$, independent of k, because $\epsilon(T)$ itself only depends on M and T, and it is independent of k. Therefore our iteration scheme can be conducted for all k, and we can construct the sequence $\{U^k\}$.

4. Convergence of U^k

In this section, we are going to prove Theorem 2.2. Taking into account that $\{U^k\}_k$ is uniformly bounded in $H^m(\Omega_T)$, we already know that there exists a subsequence of $\{U^k\}$ converging weakly in $H^m(\Omega_T)$. The next point is to prove that the sequence $\{U^k\}$ is Cauchy in $L^2(\Omega_T)$. We write

$$W^{k+1} = U^{k+1} - U^k. \tag{4.1}$$

Let us make explicit the equation satisfied by W^{k+1}. By (2.19) we have:

$$\begin{aligned} U_t^{k+1} + A(U_a + U^k + U_s)U_x^{k+1} &= F^{U^k}, \\ U_t^k + A(U_a + U^{k-1} + U_s)U_x^k &= F^{U^{k-1}}, \end{aligned}$$

and subtracting these two equations, we arrive at

$$\begin{aligned} W_t^{k+1} &+ A(U_a + U^k + U_s)W_x^{k+1} \\ &= F^{U^k} - F^{U^{k-1}} - (A(U_a + U^k + U_s) - A(U_a + U^{k-1} + U_s))U_x^k \\ &=: \widehat{F}^{W^k}. \end{aligned} \tag{4.2}$$

By direct computation, we also have:

$$\text{I.C.} \quad W^{k+1}|_{t\le 0} = 0, \qquad \text{B.C.} \quad W^{k+1}|_{x=0} = 0.$$

Let us find the explicit form of $\widehat{F}^{W^k}$. We have

$$\begin{aligned} F^{U^k} - F^{U^{k-1}} &= \phi(U^k, x) - \phi(U^{k-1}, x) - (L_{U_a+U^k+U_s}(U_a + U_s) \\ &\quad - L_{U_a+U^{k-1}+U_s}(U_a + U_s)) \\ &= \phi(U^k, x) - \phi(U^{k-1}, x) - (A(U_a + U^k + U_s) \\ &\quad - A(U_a + U^{k-1} + U_s))(U_a + U_s)_x. \end{aligned}$$

Hence

$$\begin{aligned} \widehat{F}^{W^k} = \phi(U^k, x) - \phi(U^{k-1}, x) - (A(U_a + U^k + U_s) \\ -A(U_a + U^{k-1} + U_s))(U_a + U^k + U_s)_x. \end{aligned}$$

The terms in $\widehat{F}^{W^k}$ can be written as follows:

$$\phi(U^k, x) - \phi(U^{k-1}, x) = \big(f(v^k - v^{k-1}), -f(u^k - u^{k-1}), 0\big)^t;$$

$$\begin{aligned} &A(U_a + U^k + U_s) - A(U_a + U^{k-1} + U_s) \\ &\qquad = \begin{pmatrix} u^k - u^{k-1} & 0 & 0 \\ 0 & u^k - u^{k-1} & 0 \\ h^k - h^{k-1} & 0 & u^k - u^{k-1} \end{pmatrix}, \\ &\big(A(U_a + U^k + U_s) - A(U_a + U^{k-1} + U_s)\big)(U_a + U^k + U_s)_x \\ &\qquad = (u^k - u^{k-1})(U_a + U^k + U_s)_x + (h^k - h^{k-1})\big(0, 0, (u_a + u^k + u_s)_x\big)^t; \end{aligned}$$

where $(\ ,\ ,\ )^t$ denotes the transpose of a matrix.

Using these calculations, it follows that

$$\begin{aligned}\|\widehat{F}^{W^k}\|^2_{L^2(\Omega_T)} &\le f^2\|W^k\|^2_{L^2(\Omega_T)} + \|(U_a + U^k + U_s)_x\|^2_{L^\infty(\Omega_T)}\|W^k\|^2_{L^2(\Omega_T)} \\ &\le (f^2 + C_4(M))\|W^k\|^2_{L^2(\Omega_T)}.\end{aligned}$$

Observing that equation (4.2) also has the same form as (2.19), we therefore proceed exactly as for (3.11). Noticing that the boundary data of W^{k+1} vanishes, we find

$$\begin{aligned}\|W^{k+1}\|^2_{L^2(\Omega_T)} &\le \epsilon(T)\|\widehat{F}^{W^k}\|^2_{L^2(\Omega_T)} \\ &\le \epsilon(T)(f^2 + C_4(M))\|W^k\|^2_{L^2(\Omega_T)}.\end{aligned} \tag{4.3}$$

Upon reducing T again, we can assume that

$$\epsilon(T)(f^2 + C_4(M)) \le \frac{1}{4}. \tag{4.4}$$

Hence, we have

$$\|W^{k+1}\|_{L^2(\Omega_T)} \le \frac{1}{2}\|W^k\|_{L^2(\Omega_T)} \le \cdots \le (\frac{1}{2})^k\|W^1\|_{L^2(\Omega_T)}.$$

Therefore $\{U^k\}$ is a Cauchy sequence in $L^2(\Omega_T)$, and let us call $\overline{U}$ its limit in $L^2(\Omega_T)$. We also have proven that $\{U^k\}$ is uniformly bounded in $H^m(\Omega_T)$ $(m \ge 3)$, so by $L^2 - H^m$ interpolation, the sequence $\{U^k\}$ converges strongly to $\overline{U}$ in $H^{m-1}(\Omega_T)$. Now passing to the limit in (2.19), we obtain that $\overline{U}$ is a solution of (2.18). Writing $\widetilde{U} = U_a + \overline{U}$, then $\widetilde{U}$ is the solution which we seek in Theorem 2.2. The uniqueness directly follows from (4.3).

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