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On the Neumann problem with a nonlinear boundary condition

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Abstract - We investigate the solvability of the Neumann problem with a nonlinear boundary condition. We distinguish two cases: concave and convex nonlinearity on the boundary. In the concave case we prove the existence of at least two solutions.

Key words and phrases : Neumann problem, nonlinear boundary condition, convex-concave nonlinearities, critical Sobolev exponent.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with a smooth boundary $\partial\Omega$. We consider the following Neumann problem

$$\begin{cases} -\Delta u + u & = \mu Q(x) u^{2^*-1} \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} & = \lambda u^{q-1} \text{ on } \partial\Omega, \; u > 0 \text{ on } \Omega \end{cases} \tag{1.1}$$

where $Q(x)$ is a continuous and positive function on $\bar{\Omega}$. $\lambda > 0$ and $\mu > 0$ are parameters and $2^* = \frac{2N}{N-2}$ denotes the critical Sobolev exponent. It is assumed that exponent q satisfies $1 < q < 2^{**}$, where $2^{**} = \frac{2(N-1)}{N-2}$ is a critical Sobolev exponent for the trace embedding.

In recent years problems of the form (1.1) (with a nonlinearity at the boundary condition) have attracted considerable interest. These problems originate in the studies of optimal constants for the Sobolev trace embeddings, in the theory of quasi-conformal mappings on Riemannian manifolds and reaction diffusion problems [12], [13], [15], [6]. We also refer to papers [19], [17] and [24], where further bibliographical references can be found. Our goal is to investigate problem (1.1) in two cases: (i) $1 < q < 2$ (concave case) and (ii) $2 \leq q < 2^{**}$ (convex case). For the results in the case $q = 2^{**}$, we refer to papers [9], [10] and [21]. In the case of the Dirichlet problem, the first existence results have been obtained in [4]. In this case a concave - convex perturbation appears in the right hand side of the equation. Since then, problems involving concave nonlinearities have been studied by many authors (see [16], [20], [23] and bibliographical references given there).

We point out that we are mainly interested in the existence of positive solutions of problem (1.1). This problem can be regarded as a stationary state of the corresponding evolution equation. The right hand side of equation (1.1) contains a positive reaction term and the boundary condition involves a positive flux term through the boundary. In order to obtain equilibrium the left hand side of equation (1.1) contains a term $+u$ (see also [16]). In the final part of this paper we also consider problem (1.1) without the term $+u$ on the left hand side of the equation. To obtain positive solutions in this situation, we assume that the coefficient $Q(x)$ changes sign.

Throughout this work we use standard notations. The norms in the Lebesgue spaces $L^p(\Omega)$, $1 \leq p < \infty$ are denoted by $\|\cdot\|_p$. The symbol $|A|$ stands for the Lebesgue measure of a set $A \subset \mathbb{R}^N$. The Hausdorff $N-1$ dimensional measure is denoted by the same symbol. In a given Banach space X, we denote strong convergence by " $\to$ " and weak convergence by " $\rightharpoonup$ ". The duality pairing between X and its dual space X^* is denoted by $\langle\cdot,\cdot\rangle$. Solutions of problem (1.1) are sought in the Sobolev space $H^1(\Omega)$. We recall that by $H^1(\Omega)$ we denote the usual Sobolev space equipped with norm

$$\|u\|^2 = \int_\Omega \big(|\nabla u|^2 + |u|^2\big)\, dx.$$

The paper is organized as follows. In Section 2 we give a nonexistence result in the concave case. In Sections 3 and 4 we discuss the existence of solutions in the concave case. Our approach is based on the Ekeland variational principle and the mountain-pass theorem (see [14] and [3]). Section 5 is devoted to the convex case. In the final Section 6 we investigate problem (1.1) without the term $+u$ on the left hand side of equation (1.1).

2. Nonexistence of positive solutions in the concave case

It follows from the regularity theory that solutions to problem (1.1) are continuous and positive on $\bar{\Omega}$ (see [17]). We put $Q_M = \max_{x\in\bar{\Omega}} Q(x)$ and $Q_m = \max_{x\in\partial\Omega} Q(x)$. For technical purposes we transform the unknown function u in (1.1) to transfer a parameter μ to the boundary condition.

Let $v = \frac{u}{\beta}$, where a constant $\beta > 0$ will be chosen in the following way. It is clear that v satisfies

$$\begin{cases} -\Delta v + v &= \mu\beta^{2^*-2}Q(x)v^{2^*-1} \ \text{ in } \ \Omega, \\ \frac{\partial v}{\partial \nu} &= \lambda\beta^{q-2}v^{q-1} \ \text{ on } \ \partial\Omega, \ v > 0 \text{ on } \Omega. \end{cases} \tag{2.1}$$

We choose β so that $\mu\beta^{2^*-2} = 1$. With this choice of β (2.1) becomes

$$\begin{cases} -\Delta v + v &= Q(x)v^{2^*-1} \ \text{ in } \ \Omega, \\ \frac{\partial v}{\partial \nu} &= \lambda\mu^{\frac{2-q}{2^*-2}}v^{q-1} \ \text{ on } \ \partial\Omega, \ v > 0 \text{ on } \Omega. \end{cases} \tag{2.2}$$

A solution $v \in H^1(\Omega)$ of (2.2) is understood in the weak sense

$$\int_\Omega (\nabla v \nabla \phi + v\phi)\,dx = \int_\Omega Q(x) v^{2^*-1}\phi\,dx + \lambda\mu^{\frac{2-q}{2^*-2}} \int_{\partial\Omega} v^{q-1}\phi\,dS_x \qquad (2.3)$$

for every function $\phi \in H^1(\Omega)$. An analogous definition applies to problem (1.1).

Proposition 2.1. *Suppose that* $1 < q < 2$. *If*

$$\lambda\mu^{\frac{2-q}{2^*-2}} > \frac{(2^*-2)(2-q)^{\frac{2-q}{2^*-2}}|\Omega|}{(2^*-q)^{\frac{2^*-q}{2^*-2}} Q_m^{\frac{2-q}{2^*-2}} |\partial\Omega|},$$

then problem (2.2) *does not have a solution.*

Proof. Suppose that v is a solution of (2.2). We take as a test function in (2.3) $\phi = v^{1-q}$. Since $v > 0$ on $\bar{\Omega}$, $v \in H^1(\Omega)$. We have

$$(1-q)\int_\Omega |\nabla v|^2 v^{-q}\,dx + \int_\Omega v^{2-q}\,dx = \int_\Omega Q(x)v^{2^*-q}\,dx + \lambda\mu^{\frac{2-q}{2^*-2}}\int_{\partial\Omega} dS_x. \qquad (2.4)$$

By the Young inequality we have for $\delta > 0$

$$\begin{aligned}\int_\Omega v^{2-q}\,dx &\le \frac{2-q}{2^*-q}\delta^{\frac{2^*-q}{2-q}}\int_\Omega v^{2^*-q}\,dx + \frac{(2^*-2)|\Omega|}{(2^*-q)\delta^{\frac{2^*-q}{2^*-2}}} \\ &\le \frac{2-q}{2^*-q}\delta^{\frac{2^*-q}{2-q}}\frac{1}{Q_m}\int_\Omega Q(x)v^{2^*-q}\,dx + \frac{(2^*-2)|\Omega|}{(2^*-q)\delta^{\frac{2^*-q}{2^*-2}}}.\end{aligned}$$

Choosing δ so that $\frac{2-q}{2^*-q}\delta^{\frac{2^*-q}{2-q}} = Q_m$, we derive the following estimate

$$\int_\Omega v^{2-q}\,dx \le \int_\Omega Q(x)v^{2^*-q}\,dx + \frac{(2^*-2)(2-q)^{\frac{2-q}{2^*-2}}|\Omega|}{(2^*-q)^{\frac{2^*-q}{2^*-2}}Q_m^{\frac{2-q}{2^*-2}}}.$$

Inserting this inequality into (2.4) we obtain

$$\lambda\mu^{\frac{2-q}{2^*-2}}|\partial\Omega| \le \frac{(2^*-2)(2-q)^{\frac{2-q}{2^*-2}}|\Omega|}{(2^*-q)^{\frac{2^*-q}{2^*-2}}Q_m^{\frac{2-q}{2^*-2}}}$$

and the result follows. □

Lemma 2.1. *Any solution to problem* (2.2) *with* $1 < q < 2^{**}$ *satisfies the following estimates*

$$\int_{\partial\Omega} v^{q-1}\,dS_x \le \lambda^{-1}\mu^{-\frac{2-q}{2^*-2}}\frac{(2^*-2)|\Omega|}{(2^*-1)\left[(2^*-1)Q_m\right]^{\frac{1}{2^*-2}}} \qquad (2.5)$$

and

$$Q_m \int_\Omega v^{2^*-1}\,dx \le \int_\Omega Q(x)v^{2^*-1}\,dx \le (2^*-2)\left(\frac{2}{2^*-1}\right)^{\frac{2^*-1}{2^*-2}} \frac{|\Omega|}{Q_m^{\frac{1}{2^*-2}}}. \quad (2.6)$$

Proof. Testing (2.3) with $\phi(x) = 1$ we obtain

$$\int_\Omega v\,dx = \int_\Omega Q(x)v^{2^*-1}\,dx + \lambda\mu^{\frac{2-q}{2^*-1}} \int_{\partial\Omega} v^{q-1}\,dS_x. \quad (2.7)$$

Applying the Young inequality we get

$$\int_\Omega v\,dx \le \int_\Omega Q(x)v^{2^*-1}\,dx + \frac{2^*-2}{2^*-1}|\Omega| \frac{1}{[(2^*-1)Q_m]^{\frac{1}{2^*-2}}}.$$

Combining this inequality with (2.7) we derive (2.5). To derive the estimate (2.6) we use the Young inequality in the following way: for every $\delta > 0$ we have

$$\int_\Omega v\,dx \le \frac{\delta^{2^*-1}}{(2^*-1)Q_m} \int_\Omega Q(x)v^{2^*-1}\,dx + \frac{(2^*-2)|\Omega|}{(2^*-1)\delta^{\frac{2^*-1}{2^*-2}}}.$$

We choose $\delta = \left(\frac{(2^*-1)Q_m}{2}\right)^{\frac{1}{2^*-1}}$. We then get

$$\int_\Omega v\,dx \le \frac{1}{2} \int_\Omega Q(x)v^{2^*-1}\,dx + \frac{(2^*-2)2^{\frac{1}{2^*-2}}|\Omega|}{(2^*-1)^{\frac{2^*-1}{2^*-2}} Q_m^{\frac{1}{2^*-2}}}.$$

This combined with (2.7) yields (2.6). □

In the sequel we only use estimate (2.6).

3. Local minimization (concave case)

Since the nonlinearity at the boundary condition is concave we can expect the existence of at least two distinct solutions. Both solutions will be obtained as critical points of the variational functional

$$J_{\lambda,\mu}(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{2^*} \int_\Omega Q(x)|u|^{2^*}\,dx - \frac{\lambda\mu^{\frac{2-q}{2^*-2}}}{q} \int_{\partial\Omega} |u|^q\,dS_x.$$

It is clear that this functional is of class C^1 on $H^1(\Omega)$.

Throughout this paper we frequently use the following best Sobolev constants

$$S = \inf_{H_0^1(\Omega)-\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\left(\int_\Omega |u|^{2^*}\,dx\right)^{\frac{2}{2^*}}},$$

$$S_1 = \inf_{H^1(\Omega)-\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 + u^2\right) dx}{\left(\int_\Omega |u|^{2^*}\, dx\right)^{\frac{2}{2^*}}}$$

and the best Sobolev constant for the trace embedding

$$S_q = \inf_{H^1(\Omega)-\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 + u^2\right) dx}{\left(\int_{\partial\Omega} |u|^q\, dS_x\right)^{\frac{2}{q}}}$$

for $1 < q \le 2^{**}$. Here $H_0^1(\Omega)$ is the Sobolev space obtained as the completion of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2\, dx.$$

In Proposition 3.1, below, we show that the functional $J_{\lambda,\mu}$ has a mountain-pass structure.

Proposition 3.1. *Let $1 < q < 2$. Then*

(i) for every $\mu > 0$ there exists $\lambda_\circ > 0$, $\rho > 0$ and $\kappa > 0$ such that

$$J_{\lambda,\mu}(u) \ge \kappa \quad \text{for } \|u\| = \rho \quad \text{and } 0 < \lambda < \lambda_\circ,$$

(ii) for every $\lambda > 0$ there exists $\mu_\circ > 0$, $\rho > 0$ and $\kappa > 0$ such that

$$J_{\lambda,\mu}(u) \ge \kappa \quad \text{for } \|u\| = \rho \quad \text{and } 0 < \mu < \mu_\circ.$$

Moreover, in both cases we have

$$\inf_{\|u\| \le \rho} J_{\lambda,\mu}(u) < 0. \tag{3.1}$$

Proof. By the Sobolev embedding theorems we have

$$J_{\lambda,\mu}(u) \ge \frac{1}{2}\|u\|^2 - \frac{S_1^{-\frac{2^*}{2}} Q_M}{2^*}\|u\|^{2^*} - \frac{\lambda \mu^{\frac{2-q}{2^*-2}}}{q} S_q^{-\frac{q}{2}} \|u\|^q.$$

First, we choose $\rho > 0$ so small that

$$\frac{1}{2}\|u\|^2 - \frac{S_1^{-\frac{2^*}{2}} Q_M}{2^*}\|u\|^{2^*} \ge 2\kappa$$

for $\|u\| = \rho$ and some constant $\kappa > 0$. It suffices now to choose either $\lambda_\circ > 0$ small (for a fixed $\mu > 0$) or a small $\mu_\circ > 0$ (for a fixed $\lambda > 0$) to obtain (i) and (ii). Finally, taking $t > 0$ small enough we obtain

$$J_{\lambda,\mu}(t) = \frac{t^2}{2}|\Omega| - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)\,dx - \frac{\lambda\mu^{\frac{2-q}{2^*-2}}}{q}|\partial\Omega|t^q < 0,$$

since $1 < q < 2$. This obviously implies (3.1). □

We are now in a position to establish the existence of the first solution. In the proof we use the Ekeland variational principle [14] which generates a $(PS)_c$ sequence. We recall that a C^1-functional $\Phi : X \to \mathbb{R}$ on a Banach space X satisfies the Palais - Smale condition at level $c \in \mathbb{R}$ ($(PS)_c$ condition for short) if each sequence $\{x_n\} \subset X$ such that (*) $\Phi(x_n) \to c$ and (**) $\Phi'(x_n) \to 0$ in X^* is relatively compact in X. Finally, any sequence satisfying (*) and (**) is called a Palais-Smale sequence at level c (a $(PS)_c$ sequence for short).

Theorem 3.1. *Let $1 < q < 2$ and let $\lambda_\circ$ and $\mu_\circ$ be chosen as in Proposition* 3.1*. Then*

(i) for every fixed $\mu > 0$ with $0 < \lambda < \lambda_\circ(\mu)$ problem (2.2) *admits a solution,*

(ii) for every fixed $\lambda > 0$ with $0 < \mu < \mu_\circ(\lambda)$ problem (2.2) *admits a solution.*

Proof. We only consider the case (i). It follows from Proposition 3.1 that

$$a_{\lambda\mu} = \inf_{\|u\|\le\rho} J_{\lambda,\mu}(u) < 0.$$

By the Ekeland variational principle [14] there exists a $(PS)_{a_{\lambda\mu}}$ sequence $\{u_n\} \subset H^1(\Omega)$. Since $\{u_n\}$ is bounded in $H^1(\Omega)$, we may assume, up to a subsequence, that $u_n \rightharpoonup u$ in $H^1(\Omega)$ and $L^{2^*}(\Omega)$, $u_n \to u$ in $L^2(\Omega)$ and $L^q(\partial\Omega)$. It is clear that u is a weak solution of (2.2). We then have

$$\begin{aligned} a_{\lambda\mu} &\ge \liminf_{n\to\infty}\left[J_{\lambda,\mu}(u_n) - \frac{1}{2^*}\langle J'_{\lambda,\mu}(u_n), u_n\rangle\right] \\ &\ge \frac{1}{N}\int_\Omega \big(|\nabla u|^2 + u^2\big)\,dx - \big(\frac{1}{q} - \frac{1}{2^*}\big)\lambda\mu^{\frac{2-q}{2^*-2}}\int_{\partial\Omega}|u|^q\,dS_x \\ &= J_{\lambda,\mu}(u) - \frac{1}{2^*}\langle J'_{\lambda,\mu}(u), u\rangle = J_{\lambda,\mu}(u) \ge a_{\lambda\mu}. \end{aligned}$$

Thus $J_{\lambda,\mu}(u) = a_{\lambda\mu}$ and u is a solution belonging to the interior of the ball $B(0,\rho)$. Since $|u|$ is also a minimizer we may assume by the Harnack inequality that $u > 0$ on Ω. □

4. Existence of a second solution (concave case)

In what follows we denote by $u_{\lambda\mu}$ a positive solution to problem (2.2) found in Theorem 3.1.

Proposition 4.1. *Suppose that $1 < q < 2$ and let $\lambda_\circ$ and $\mu_\circ$ be chosen as in Proposition 3.1. If $u = u_{\lambda\mu}$ and $u = 0$ are the only critical points of $J_{\lambda,\mu}$, then $(PS)_c$ condition holds for*

$$c < \tilde{c} = J_{\lambda,\mu}(u_{\lambda\mu}) + \min\Big(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}} \Big). \tag{4.1}$$

Proof. (We obviously assume that either (i) or (ii) of Proposition 3.1 holds.) Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_c$ sequence for $J_{\lambda,\mu}$, with $c < \tilde{c}$, that is, $J_{\lambda,\mu}(u_n) \to c$ and $J'_{\lambda,\mu}(u_n) \to 0$ in $H^{-1}(\Omega)$. First, we show that $\{u_n\}$ is bounded in $H^1(\Omega)$. We have

$$\begin{aligned} c + o(1) + o(\|u_n\|) &= J_{\lambda,\mu}(u_n) - \frac{1}{2}\langle J'_{\lambda,\mu}(u_n), u_n\rangle \\ &= \frac{1}{N}\int_\Omega |u_n|^{2^*}\,dx + \big(\frac{1}{2} - \frac{1}{q}\big)\lambda\mu^{\frac{2-q}{2^*-2}} \int_{\partial\Omega} |u|^q\,dS_x. \end{aligned}$$

By the Sobolev trace embedding theorem we obtain

$$\frac{1}{N}\int_\Omega |u_n|^{2^*}\,dx \le \big(\frac{1}{q} - \frac{1}{2}\big)\lambda\mu^{\frac{2-q}{2^*-2}} S_q^{-\frac{q}{2}}\|u_n\|^q + c + o(1) + o(\|u_n\|).$$

This inequality combined with the fact that $J_{\lambda,\mu}(u_n) \to c$ gives

$$\begin{aligned} \frac{1}{2}\int_\Omega \big(|\nabla u_n|^2 + u_n^2\big)\,dx &\le c + 1 + \frac{1}{2^*}\int_\Omega Q(x)|u_n|^{2^*}\,dx \\ &+ \frac{\lambda\mu^{\frac{2-q}{2^*-2}}}{q}\int_{\partial\Omega} |u_n|^q\,dS_x \\ &\le c + 1 + C(\|u_n\| + \|u_n\|^q) \end{aligned}$$

for $n \ge n_\circ$, where $C > 0$ is a constant independent of n. Since $1 < q < 2$, we see that $\{u_n\}$ is bounded in $H^1(\Omega)$. We may assume, up to a subsequence, that $u_n \rightharpoonup u$ in $H^1(\Omega), L^{2^*}(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$, $L^q(\Omega)$. By the P.L. Lions' concentration - compactness principle (see [18]) there exist at most countable set of points $\{x_j\} \subset \bar{\Omega}$, and positive numbers $\{\mu_j\}$, $\{\nu_j\}$, $j \in J$, such that

$$|u_n|^{2^*}\,dx \rightharpoonup d\nu = |u|^{2^*}\,dx + \sum_{j\in J}\nu_j\delta_{x_j}$$

and

$$|\nabla u_n|^2\,dx \rightharpoonup d\mu \ge |\nabla u|^2\,dx + \sum_{j\in J}\mu_j\delta_{x_j},$$

in the sense of measure, where δ_{x_j} denotes the Dirac measures assigned to point x_j. Moreover, the numbers μ_j and ν_j satisfy the following inequalities

$$S\nu_j^{\frac{2}{2^*}} \le \mu_j \text{ if } x_j \in \Omega, \tag{4.2}$$

and

$$\frac{S}{2^{\frac{2}{N}}}\nu_j^{\frac{2}{2^*}} \le \mu_j \text{ if } x_j \in \partial\Omega. \tag{4.3}$$

Testing $J'_{\lambda,\mu}(u_n) \to 0$ in $H^{-1}(\Omega)$ with a family of functions φ_δ, $\delta > 0$, concentrating at x_j as $\delta \to 0$, we derive

$$\mu_j \le Q(x_j)\nu_j, \quad j \in J. \tag{4.4}$$

If $\nu_j > 0$, then in the case $x_j \in \Omega$, (4.2) and (4.4) imply

$$\nu_j \ge \frac{S^{\frac{N}{2}}}{Q(x_j)^{\frac{N}{2}}}. \tag{4.5}$$

On the other hand, if $x_j \in \partial\Omega$, then by (4.3) and (4.4) we obtain

$$\nu_j \ge \frac{S^{\frac{N}{2}}}{2Q(x_j)^{\frac{N}{2}}}. \tag{4.6}$$

By the Brezis - Lieb lemma (see [7]) we get for $v_n = u_n - u$

$$\int_\Omega Q(x)|u_n|^{2^*}\,dx = \int_\Omega Q(x)|u|^{2^*}\,dx + \int_\Omega Q(x)|v_n|^{2^*}\,dx + o(1).$$

We also have

$$\int_\Omega |\nabla u_n|^2\,dx = \int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla v_n|^2\,dx + o(1).$$

These relations are used to rewrite

$$J_{\lambda,\mu}(u_n) = c + o(1) \text{ and } J'_{\lambda,\mu}(u_n) = o(1) \text{ in } H^{-1}(\Omega)$$

in the following form

$$J_{\lambda,\mu}(u) + \frac{1}{2}\int_\Omega |\nabla v_n|^2\,dx - \frac{1}{2^*}\int_\Omega Q(x)|v_n|^{2^*}\,dx = c + o(1) \tag{4.7}$$

and

$$\begin{aligned}\int_\Omega \big(|\nabla u|^2 + u^2\big)\,dx \quad &- \quad \int_\Omega Q(x)|u|^{2^*}\,dx - \lambda\mu^{\frac{2-q}{2^*-2}}\int_{\partial\Omega} |u|^q\,dS_x \\ &+ \quad \int_\Omega \big(|\nabla v_n|^2 - Q(x)|v_n|^{2^*}\big)\,dx = o(1).\end{aligned} \tag{4.8}$$

Since u is a weak solution of (2.2) we deduce from (4.8)

$$\int_\Omega |\nabla v_n|^2\,dx = \int_\Omega Q(x)|v_n|^{2^*}\,dx + o(1).$$

Substituting this into (4.7) we get

$$J_{\lambda,\mu}(u) + \frac{1}{N}\int_\Omega |\nabla v_n|^2\,dx = c + o(1). \tag{4.9}$$

From (4.2)–(4.6) we obtain the following inequalities

$$\mu_j \geq \frac{S^{\frac{N}{2}}}{Q_M^{\frac{N-2}{2}}} \quad \text{if } \nu_j > 0,\ x_j \in \Omega$$

and

$$\mu_j \geq \frac{S^{\frac{N}{2}}}{2Q_m^{\frac{N-2}{2}}} \quad \text{if } \nu_j > 0,\ x_j \in \partial\Omega.$$

To complete the proof, we show that if either $u = 0$ or $u = u_{\lambda\mu}$, then $\mu_j = \nu_j = 0$ for all $j \in J$. Arguing by contradiction, we would have $\nu_j > 0$ for some $j \in J$. Then by (4.9) we would have in the first case $\frac{S^{\frac{N}{2}}}{Q_M^{\frac{N-2}{2}}} \leq c$ if $x_j \in \Omega$ and $\frac{S^{\frac{N}{2}}}{2Q_m^{\frac{N-2}{2}}} \leq c$ if $x_j \in \partial\Omega$. This shows that

$$c \geq \min\left(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}\right)$$

which is impossible. If $u = u_{\lambda\mu}$ we would have

$$J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}} \leq c \quad \text{if } x_j \in \Omega$$

and

$$J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}} \leq c \quad \text{if } x_j \in \partial\Omega.$$

This yields

$$J_{\lambda,\mu}(u_{\lambda\mu}) + \min\left(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}\right) \leq c$$

which gives a contradiction. □

A second solution will be obtained by the mountain-pass principle (see [3]). First we consider the case $Q_M \leq 2^{\frac{2}{N-2}} Q_m$. Then $(PS)_c$ condition holds for

$$c \leq J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}.$$

For simplicity we assume that $Q_m = Q(0)$, $0 \in \partial\Omega$. To estimate the energy level of $J_{\lambda,\mu}$ we use a family of instantons

$$U_{\epsilon,y}(x) = \frac{c_N \epsilon^{\frac{N-2}{2}}}{\left(\epsilon^2 + |x-y|^2\right)^{\frac{N-2}{2}}},\ \epsilon > 0,\ y \in \mathbb{R}^N.$$

It is known that $U_{\epsilon,y}$ satisfies the equation

$$-\Delta u = |u|^{2^*-2} u \ \text{ in } \mathbb{R}^N$$

and

$$\|U_{\epsilon,y}\|_{2^*}^{2^*} = \|\nabla U_{\epsilon,y}\|_2^2 = S^{\frac{N}{2}}.$$

If $y = 0$ we put $U_\epsilon = U_{\epsilon,0}$. If $H(0)$ denotes the mean curvature of $\partial\Omega$ at 0, then the following estimates hold

$$\frac{\int_\Omega |\nabla U_\epsilon|^2\,dx}{\left(\int_\Omega U_\epsilon^{2^*}\,dx\right)^{\frac{2}{2^*}}} \leq \begin{cases} \frac{S}{2^{\frac{2}{N}}} - A_N H(0)\epsilon\log\frac{1}{\epsilon} + O(\epsilon) & \text{if } N=3, \\ \frac{S}{2^{\frac{2}{N}}} - A_N H(0)\epsilon + O\left(\epsilon^2\log\frac{1}{\epsilon}\right) & \text{if } N=4, \\ \frac{S}{2^{\frac{2}{N}}} - A_N H(0)\epsilon + O\left(\epsilon^2\right) & \text{if } N\geq 5, \end{cases} \tag{4.10}$$

where $A_N > 0$ is a constant depending on N (see [1], [2], [22]).

Proposition 4.2. *Suppose that $1 < q < 2$ and $N \geq 5$ and let $\lambda_\circ, \mu_\circ$ be constants chosen in Proposition 3.1. Assume that either (i) or (ii) of Theorem 3.1 holds. Suppose that $Q_M \leq 2^{\frac{2}{N-2}} Q_m$, $Q_m = Q(0)$, $0 \in \partial\Omega$ and $H(0) > 0$ and that*

$$|Q(x) - Q(0)| = o(|x|) \ \text{ for } x \text{ near } 0. \tag{4.11}$$

Then

$$\max_{t\geq 0} J_{\lambda,\mu}(u_{\lambda\mu} + tU_\epsilon) < J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}$$

for $\epsilon > 0$ sufficiently small.

Proof. We use some ideas from the proof of Lemma 2.5 in [20]. We use the following inequality: given $q > 2$ and $\kappa \in (1, q-1)$ there exists a constant $C > 0$ such that

$$(s+t)^q \geq s^q + t^q + qs^{q-1}t + qst^{q-1} - Ct^\kappa s^{q-\kappa}$$

for every $s \geq 0$ and $t \geq 0$. Applying this inequality to the integral

$$\int_\Omega Q(x)\big(u_{\lambda\mu} + tU_\epsilon\big)^{2^*}\, dx \ \text{ with } \ \kappa = \frac{N+1}{N-2}$$

we get

$$\begin{aligned}
J_{\lambda,\mu}(u_{\lambda\mu} + tU_\epsilon) \ &\leq \ \frac{1}{2}\int_\Omega |\nabla u_{\lambda\mu}|^2\, dx + \frac{t^2}{2}\int_\Omega |\nabla U_\epsilon|^2\, dx + t\int_\Omega \nabla u_{\lambda\mu}\nabla U_\epsilon\, dx \\
&+ \ \frac{1}{2}\int_\Omega u_{\lambda\mu}^2\, dx + \frac{t^2}{2}\int_\Omega U_\epsilon^2\, dx + t\int_\Omega U_\epsilon u_{\lambda\mu}\, dx \\
&- \ \frac{t^{2^*}}{2^*}\int_\Omega Q(x)U_\epsilon^{2^*}\, dx - \frac{1}{2^*}\int_\Omega Q(x)u_{\lambda\mu}^{2^*}\, dx \\
&- \ t^{2^*-1}\int_\Omega Q(x)U_\epsilon^{2^*-1}u_{\lambda\mu}\, dx - t\int_\Omega Q(x)u_{\lambda\mu}^{2^*-1}U_\epsilon\, dx \\
&+ \ Ct^{\frac{N+1}{N-2}}\int_\Omega Q(x)u_{\lambda\mu}^{\frac{N-1}{N-2}}U_\epsilon^{\frac{N+1}{N-2}}\, dx \\
&- \ \frac{\lambda\mu^{\frac{2-q}{2^*-2}}}{q}\int_{\partial\Omega}\big(u_{\lambda\mu} + tU_\epsilon\big)^q\, dS_x \\
&= \ J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{\lambda\mu^{\frac{2-q}{2^*-2}}}{q}\int_{\partial\Omega}\Big[u_{\lambda\mu}^q - \big(u_{\lambda\mu} + tU_\epsilon\big)^q\Big]\, dS_x \\
&+ \ \lambda\mu^{\frac{2-q}{2^*-2}}t\int_{\partial\Omega} u_{\lambda\mu}^{q-1}U_\epsilon\, dS_x \\
&+ \ \frac{t^2}{2}\int_\Omega \big(|\nabla U_\epsilon|^2 + U_\epsilon^2\big)\, dx - t^{2^*-1}\int_\Omega Q(x)U_\epsilon^{2^*-1}u_{\lambda\mu}\, dx \\
&+ \ Ct^{\frac{N+1}{N-2}}\int_\Omega Q(x)u_{\lambda\mu}^{\frac{N-1}{N-2}}U_\epsilon^{\frac{N+1}{N-2}}\, dx - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)U_\epsilon^{2^*}\, dx.
\end{aligned}$$

We now use the inequality

$$\frac{1}{q}\big(u_{\lambda\mu} + tU_\epsilon\big)^q - \frac{u_{\lambda\mu}^q}{q} - tu_{\lambda\mu}^{q-1}U_\epsilon \geq 0$$

to obtain

$$\begin{aligned}
J_{\lambda,\mu}(u_{\lambda\mu} + tU_\epsilon) \ &\leq \ J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{t^2}{2}\int_\Omega \big(|\nabla U_\epsilon|^2 + U_\epsilon^2\big)\, dx \\
&- \ \frac{t^{2^*}}{2^*}\int_\Omega Q(x)U_\epsilon^{2^*}\, dx - t^{2^*-1}\int_\Omega Q(x)U_\epsilon^{2^*-1}u_{\lambda\mu}\, dx \\
&+ \ Ct^{\frac{N+1}{N-2}}\int_\Omega u_{\lambda\mu}^{\frac{N-1}{N-2}}U_\epsilon^{\frac{N+1}{N-2}}\, dx.
\end{aligned} \tag{4.12}$$

To proceed further we need the following estimate

$$\int_\Omega Q(x)u_{\lambda\mu}^{\frac{N-1}{N-2}}U_\epsilon^{\frac{N+1}{N-2}}\, dx \leq C\epsilon^{\frac{N-1}{2}}$$

for some constant $C > 0$, independent of $\epsilon > 0$. Combining this estimate with (4.12) we obtain

$$\begin{aligned} J_{\lambda,\mu}(u_{\lambda\mu} + tU_\epsilon) &\leq J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{t^2}{2}\int_\Omega \big(|\nabla U_\epsilon|^2 + U_\epsilon^2\big)\,dx \\ &- \frac{t^{2^*}}{2^*}\int_\Omega Q(x)U_\epsilon^{2^*}\,dx + \Phi_\epsilon(t), \end{aligned} \tag{4.13}$$

where

$$\Phi_\epsilon(t) = Ct^{\frac{N+1}{N-2}}\epsilon^{\frac{N-1}{2}}.$$

We put

$$\Psi_\epsilon(t) = \frac{t^2}{2}\int_\Omega \big(|\nabla U_\epsilon|^2 + U_\epsilon^2\big)\,dx - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)U_\epsilon^{2^*}\,dx + \Phi_\epsilon(t).$$

It obvious that $\lim_{t\to 0}\Psi_\epsilon(t) = 0$ and $\lim_{t\to\infty}\Psi_\epsilon(t) = -\infty$. Hence there exists $t_\epsilon > 0$ such that $\Psi_\epsilon(t_\epsilon) = \max_{t\geq 0}\Psi_\epsilon(t)$. We now show that there exist numbers $0 < T_1 < T_2$ such that $T_1 \leq t_\epsilon \leq T_2$ for every $\epsilon > 0$ sufficiently small. By Proposition 3.1 we have

$$0 < \kappa \leq \max_{t\geq 0} J_{\lambda,\mu}(u_{\lambda\mu} + tU_\epsilon) \leq J_{\lambda,\mu}(u_{\lambda\mu}) + \Psi_\epsilon(t_\epsilon) < \Psi_\epsilon(t_\epsilon).$$

This shows that $t_\epsilon \not\to 0$ as $\epsilon \to 0$. Therefore there exists $0 < T_1$ such that $0 < T_1 \leq t_\epsilon$ for $\epsilon > 0$ small. To find an upper bound for t_ϵ we argue by contradiction. Assume that $t_{\epsilon_n} \to \infty$ for some sequence $\epsilon_n \to 0$. Then $\Psi'_{\epsilon_n}(t_{\epsilon_n}) = 0$, so

$$t_{\epsilon_n}\int_\Omega \big(|\nabla U_{\epsilon_n}|^2 + U_{\epsilon_n}^2\big)\,dx = t_{\epsilon_n}^{\frac{N+2}{N-2}}\int_\Omega Q(x)U_{\epsilon_n}^{2^*}\,dx - C\frac{N+1}{N-2}t_{\epsilon_n}^{\frac{3}{N-2}}\epsilon_n^{\frac{N-2}{2}}.$$

We rewrite this relation as

$$t_{\epsilon_n}^{-\frac{4}{N-2}}\int_\Omega \big(|\nabla U_{\epsilon_n}|^2 + U_{\epsilon_n}^2\big)\,dx = \int_\Omega Q(x)U_{\epsilon_n}^{2^*}\,dx - C\frac{N+1}{N-2}t_{\epsilon_n}^{-\frac{N-1}{N-2}}\epsilon_n^{\frac{N-2}{2}}.$$

It is clear that the above relation cannot be satisfied for large n, as the left-hand side tends to 0 and $\int_\Omega U_{\epsilon_n}^{2^*}\,dx \not\to 0$ as $n \to \infty$. From (4.13) we derive the following estimate

$$J_{\lambda,\mu}(u_{\lambda\mu} + tU_\epsilon) \leq J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{\left(\int_\Omega \big(|\nabla U_\epsilon|^2 + U_\epsilon^2\big)\,dx\right)^{\frac{N}{2}}}{N\left(\int_\Omega Q(x)U_\epsilon^{2^*}\,dx\right)^{\frac{N-2}{2}}} + \Phi_\epsilon(t).$$

Assumption (4.11) implies the following expansion

$$\int_\Omega Q(x)U_\epsilon^{2^*}\,dx = Q_m\int_\Omega U_\epsilon^{2^*}\,dx + o(\epsilon).$$

We now observe that $\Phi_\epsilon(t_\epsilon) = o(\epsilon)$ and $\int_\Omega U_\epsilon^2\,dx = O(\epsilon^2)$ for $N \geq 5$. Therefore the last two relations combined with (4.10) give

$$\max_{t\geq o} J_{\lambda,\mu}(u_{\lambda\mu} + tU_\epsilon) < J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}.$$

□

To find a second solution we choose $w \in H^1(\Omega)$ such that $w \in H^1(\Omega)$ and $\|w\| > \rho$ and set

$$\Gamma = \{\gamma \in C([0,1], H^1(\Omega)) \,|\, \gamma(0) = 0,\, \gamma(1) = w\}.$$

Theorem 4.1. *Let $N \geq 5$ and $0 < q < 2$. Suppose that the assumptions of Proposition 4.2 are satisfied. Then in both cases (i) and (ii) (see Proposition 3.1) problem (2.2) admits a second solution.*

Proof. Suppose that 0 and $u_{\lambda\mu}$ are the unique critical points of $J_{\lambda,\mu}$. Applying the mountain-pass principle we derive the existence of a critical point v satisfying $J_{\lambda,\mu}(v) \geq \kappa > 0$ which is obviously distinct from 0 and $u_{\lambda\mu}$. This gives a contradiction. □

We now consider the case $Q_M > 2^{\frac{2}{N-2}} Q_m$. In this case we have

$$\tilde{c} = J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}.$$

For the ease of notation we set

$$K_N(|\Omega|, Q_m) = |\Omega|\left(\frac{1}{2} + \frac{C_N}{Q_m^{\frac{1}{2^*-2}}}\right),$$

where $C_N = (2^* - 2)\left(\frac{2}{2^*-1}\right)^{\frac{2^*-1}{2^*-2}}$.

Lemma 4.1. *Let $u_{\lambda\mu}$ be a solution of (2.2) from Theorem 3.1. Suppose that (i) and (ii) of Proposition 3.1 hold. Then*

$$\max_{t\geq 0} J_{\lambda,\mu}(u_{\lambda\mu} + t) \leq J_{\lambda,\mu}(u_{\lambda\mu})$$
$$+ \max\left[K_N(|\Omega|, Q_m), \frac{1}{N}\left(2K_N(|\Omega|, Q_m)\right)^{\frac{2^*}{2^*-2}}\left(\int_\Omega Q(x)\,dx\right)^{-\frac{2}{2^*-2}}\right].$$

Proof. We have

$$\begin{aligned}
J_{\lambda,\mu}(u_{\lambda\mu}+t) &= \frac{1}{2}\int_\Omega |\nabla u_{\lambda\mu}|^2\,dx + \frac{1}{2}\int_\Omega u_{\lambda\mu}^2\,dx + \frac{t^2}{2}|\Omega| + t\int_\Omega u_{\lambda\mu}\,dx \\
&- \frac{1}{2^*}\int_\Omega Q(x)(u_{\lambda\mu}+t)^{2^*}\,dx - \frac{\lambda\mu^{\frac{2-q}{2^*-2}}}{q}\int_{\partial\Omega}(u_{\lambda\mu}+t)^q\,dS_x \\
&= J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{1}{2^*}\int_\Omega Q(x)u_{\lambda\mu}^{2^*}\,dx \\
&+ t\int_\Omega Q(x)u_{\lambda\mu}^{2^*-1}\,dx - \frac{1}{2^*}\int_\Omega Q(x)(u_{\lambda\mu}+t)^{2^*}\,dx \\
&+ \frac{1}{q}\lambda\mu^{\frac{2-q}{2^*}}\int_{\partial\Omega} u_{\lambda\mu}^q\,dS_x + t\lambda\mu^{\frac{2-q}{2^*-2}}\int_{\partial\Omega} u_{\lambda\mu}^{q-1}\,dS_x \\
&- \frac{\lambda\mu^{\frac{2-q}{2^*-2}}}{q}\int_{\partial\Omega}(u_{\lambda\mu}+t)^q\,dS_x + \frac{t^2}{2}|\Omega| \\
&\leq J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{t^2}{2}|\Omega| + \frac{1}{2^*}\int_\Omega Q(x)u_{\lambda\mu}^{2^*}\,dx \\
&- \frac{1}{2^*}\int_\Omega Q(x)(u_{\lambda\mu}+t)^{2^*}\,dx + t\int_\Omega Q(x)u_{\lambda\mu}^{2^*-1}\,dx.
\end{aligned}$$

We now use the following inequality

$$(u_{\lambda\mu}+t)^{2^*} - u_{\lambda\mu}^{2^*} \geq t^{2^*}$$

for all $t \geq 0$ and estimate (2.6) (see Lemma 2.1). So

$$J_{\lambda,\mu}(u_{\lambda\mu}+t) \leq J_{\lambda,\mu}(u_{\lambda\mu}) + \frac{t^2}{2}|\Omega| + t\frac{|\Omega|C_N}{Q_m^{\frac{1}{2^*-2}}} - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)\,dx,$$

where $C_N = (2^*-2)\left(\frac{2}{2^*-1}\right)^{\frac{2^*-1}{2^*-2}}$. Let

$$g(t) = \frac{t^2}{2}|\Omega| + t\frac{|\Omega|C_N}{Q_m^{\frac{1}{2^*-2}}} - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)\,dx.$$

If $0 \leq t \leq 1$, then $g(t) \leq \frac{|\Omega|}{2} + \frac{|\Omega|C_N}{Q_m^{\frac{1}{2^*-2}}}$. If $t \geq 1$, then

$$\begin{aligned}
g(t) &\leq \max_{t\geq 1} g(t) \leq \max_{t\geq 0}\left[\left(\frac{1}{2} + \frac{C_N}{Q_m^{\frac{1}{2^*-2}}}\right)|\Omega|t^2 - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)\,dx\right] \\
&= \frac{1}{N}\left[2|\Omega|\left(\frac{1}{2} + \frac{C_N}{Q_m^{\frac{1}{2^*-2}}}\right)\right]^{\frac{2^*}{2^*-2}}\left(\int_\Omega Q(x)\,dx\right)^{-\frac{2}{2^*-2}}
\end{aligned}$$

and the assertion of Lemma follows. □

Theorem 4.2. *Let $0 < q < 2$ and $Q_M > 2^{\frac{2}{N-2}} Q_m$. Suppose that*

$$\max\left[K_N(|\Omega|, Q_m), \frac{1}{N}\Big(2K_N(|\Omega|, Q_m)\Big)^{\frac{2^*}{2^*-2}}\left(\int_\Omega Q(x)\,dx\right)^{-\frac{2}{2^*-2}}\right] < \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}. \tag{4.14}$$

Then in both cases (i) and (ii) (see Proposition 3.1*) problem* (2.2) *admits a second solution.*

It is clear that condition (4.14) is satisfied for sets of small measure.

Similarly, the inequality

$$\max\left[K_N(|\Omega|, Q_m), \frac{1}{N}\Big(2K_N(|\Omega|, Q_m)\Big)^{\frac{2^*}{2^*-2}}\left(\int_\Omega Q(x)\,dx\right)^{-\frac{2}{2^*-2}}\right] < \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}$$

guarantees the existence of a second solution of problem (2.2) in the case $Q_M \le 2^{\frac{2}{N-2}} Q_m^{\frac{N-2}{2}}$.

Remark 4.1. According to Theorems 3.2, 4.3 and 4.5 for every $\mu > 0$ there exists $\lambda_\circ(\mu) > 0$ such that problem (2.2) admit solutions $\bar{w}_{\lambda\mu}$ and $\tilde{w}_{\lambda\mu}$ satisfying $J_{\lambda,\mu}(\bar{w}_{\lambda\mu}) < 0$ and $J_{\lambda,\mu}(\tilde{w}_{\lambda\mu}) > 0$, respectively. Then $\bar{u}_{\lambda\mu} = \beta\bar{w}_{\lambda\mu}$ and $\tilde{u}_{\lambda\mu} = \beta\tilde{w}_{\lambda\mu}$ are solutions of problem (1.1). It then follows from estimate (2.6) that

$$\int_\Omega \bar{u}_{\lambda\mu}^{2^*-1}\,dx \le \beta^{2^*-1} C_N |\Omega| Q_m^{-\frac{2^*-1}{2^*-2}} \quad \text{and} \quad \int_\Omega \tilde{u}_{\lambda\mu}^{2^*-1}\,dx \le \beta^{2^*-1} C_N |\Omega| Q_m^{-\frac{2^*-1}{2^*-2}}.$$

Moreover, if $\mu \to \infty$, then $\beta \to 0$ (and $\lambda_\circ(\mu) \to 0$). This means that $\bar{u}_{\lambda\mu} \to 0$ and $\tilde{u}_{\lambda\mu} \to 0$ in $L^{\frac{N+2}{N-2}}(\Omega)$ as $\mu \to \infty$.

5. Convex case ($2 \le q < 2^{**}$)

In this case the dependence of equation (1.1) on a parameter $\mu > 0$ is irrelevant. Hence we consider the problem

$$\begin{cases} -\Delta u + u &= Q(x)u^{2^*-1} \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \lambda u^{q-1} \text{ on } \partial\Omega,\ u > 0 \text{ on } \Omega. \end{cases} \tag{5.1}$$

We distinguish two cases: (i) $q = 2$ and (ii) $2 < q < 2^{**}$. Solutions in both cases will be obtained through the mountain-pass principle applied to the functional

$$J_{\lambda,q}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\,dx - \frac{\lambda}{q}\int_{\partial\Omega} |u|^q\,dS_x.$$

We commence with the case (i). We note that if u is a solution of problem (5.1) then u is continuous and positive on $\bar\Omega$. Testing (5.1) with $\phi = \frac{1}{u}$ we obtain the following result:

Proposition 5.1. *Let $q = 2$. Then problem (5.1) does not admit a solution for $\lambda > \frac{|\Omega|}{|\partial\Omega|}$.*

Proposition 5.2. *Let $q = 2$. Then there exist constants $\Lambda_\circ > 0$, $\kappa > 0$ and $\rho > 0$ such that*

$$J_{\lambda,2}(u) \geq \kappa \;\; \textit{for} \;\; \|u\| = \rho \;\; \textit{and} \;\; 0 < \lambda < \Lambda_\circ. \tag{5.2}$$

The proof is similar to that of Proposition 3.1 and is omitted.

Proposition 5.3. *Let $q = 2$ and $0 < \lambda < \Lambda_\circ$. Then $(PS)_c$ condition for $J_{\lambda,2}$ holds for*

$$c < \min\left(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}\right). \tag{5.3}$$

Proof. Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_c$ sequence for $J_{\lambda,2}$. Then

$$\begin{aligned} c + 1 + o(\|u_n\|) &= J_{\lambda,2}(u_n) - \frac{1}{2^*}\langle J'_{\lambda,2}(u_n), u_n\rangle = \frac{1}{N}\int_\Omega \left(|\nabla u_n|^2 + u_n^2\right) dx \\ &- \left(\frac{1}{q} - \frac{1}{2^*}\right)\lambda \int_{\partial\Omega} |u_n|^2\,dS_x. \end{aligned}$$

Using the trace Sobolev embedding theorem and taking $\Lambda_\circ$ smaller, if necessary, we derive from the above relation that $\{u_n\}$ is bounded in $H^1(\Omega)$. We may assume that, up to a subsequence, $u_n \rightharpoonup u$ in $H^1(\Omega)$, $L^{2^*}(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$, $L^2(\partial\Omega)$. We now apply the P.L. Lions' concentration - compactness principle. So there exist at most countable sets of points $\{x_j\} \subset \bar\Omega$ and positive $\{\nu_j\}$, $\{\mu_j\}$, $j \in J$, such that

$$|u_n|^{2^*} \rightharpoonup d\nu = |u|^{2^*}\,dx + \sum_{j\in J} \nu_j \delta_{x_j}$$

and

$$|\nabla u_n|^2 \rightharpoonup d\mu \geq |\nabla u|^2\, dx + \sum_{j\in J} \mu_j \delta_{x_j}$$

where ν_j and μ_j satisfy (4.2),(4.3) and (4.4). Assuming that $\nu_j > 0$ for some $j \in J$, we get as in the proof of Proposition 4.1, estimates (4.5) and (4.6). Hence

$$\begin{aligned} c &= \lim_{n\to\infty} \left[J_{\lambda,2}(u_n) - \frac{1}{2}\langle J'_{\lambda,2}(u_n), un\rangle \right] \\ &= \frac{1}{N}\left(\int_\Omega Q(x)|u|^{2^*}\, dx + \sum_{j\in J} \nu_j Q(x_j) \right) \\ &\geq \min\left(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}} \right), \end{aligned}$$

which is impossible. □

We can now formulate the following existence result:

Theorem 5.1. *Let $N \geq 5$, $q = 2$ and $Q_M \leq 2^{\frac{2}{N-2}} Q_m$. Suppose that $H(0) > 0$, $Q_m = Q(0)$, $0 \in \partial\Omega$ and that Q satisfies* (4.11)*. Then for every $0 < \lambda < \Lambda_\circ$ there exists a solution to problem* (5.1)*.*

Proof. In order to apply the mountain-pass principle, we have to show that

$$\max_{t\geq 0} J_{\lambda,2}(tU_\epsilon) < \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}.$$

This relation is a consequence of asymptotic estimates (4.10) and assumption (4.11). □

We now consider the case $2 < q < 2^{**}$.

Proposition 5.4. *Let $2 < q < 2^{**}$ and $\lambda > 0$. Then there exist constants $\rho > 0$ and $\kappa > 0$ such that*

$$J_{\lambda,q}(u) \geq \kappa \ \ \textit{for} \ \ \|u\| = \rho. \tag{5.4}$$

Moreover the $(PS)_c$ condition holds for

$$c < \min\left(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}} \right). \tag{5.5}$$

Proof. By the Sobolev embedding theorems, we obtain

$$\begin{aligned} J_{\lambda,q}(u) &\geq \frac{1}{2}\int_\Omega \left(|\nabla u|^2+u^2\right)dx - \frac{S_1^{-\frac{2^*}{2}}}{2^*}\left(\int_\Omega \left(|\nabla u|^2+u^2\right)dx\right)^{\frac{2^*}{2}} \\ &- \frac{\lambda S_q^{-\frac{q}{2}}}{q}\left(\int_\Omega \left(|\nabla u|^2+u^2\right)dx\right)^{\frac{q}{2}}. \end{aligned}$$

Taking $\rho>0$ small enough (5.4) follows. Let $\{u_n\}\subset H^1(\Omega)$ be a $(PS)_c$ sequence with c satisfying (5.5). We have

$$\begin{aligned} c+o(1)+o(\|u_n\|) &= J_{\lambda,q}(u_n)-\frac{1}{2}\langle J'_{\lambda,q}(u_n),u_n\rangle = \frac{1}{N}\int_\Omega Q(x)|u_n|^{2^*}\,dx \\ &+ \left(\frac{1}{2}-\frac{1}{q}\right)\lambda\int_{\partial\Omega}|u_n|^q dS_x. \end{aligned}$$

Since $2<q<2^{**}$, we see that

$$\int_\Omega Q(x)|u_n|^{2^*}\,dx+\int_{\partial\Omega}|u_n|^q\,dS_x \leq C(1+\|u_n\|)$$

for some constant $C>0$ independent of n. On the other hand we have

$$J_{\lambda,q}(u_n)-\frac{1}{2^*}\langle J'_{\lambda,q}(u_n),u_n\rangle = \frac{1}{N}\int_\Omega \left(|\nabla u_n|^2+u_n^2\right)dx+\left(\frac{1}{2^*}-\frac{1}{q}\right)\lambda\int_{\partial\Omega}|u_n|^q\,dS_x.$$

From the last two relations it is easy to deduce that $\{u_n\}$ is bounded in $H^1(\Omega)$. Finally, the $(PS)_c$ condition follows from the concentration - compactness principle. □

Theorem 5.2. *Let $N\geq 5$, $2<q<2^{**}$, $H(0)>0$, $0\in\partial\Omega$, $Q_m=Q(0)$ and $Q_M\leq 2^{\frac{2}{N-2}}Q_m$. Moreover, assume that Q satisfies* (4.11)*. Then problem* (5.1) *has a solution for every $\lambda>0$.*

Proof. Using the asymptotic estimates (4.10) one can show that

$$\max_{t\geq 0} J_{\lambda,q}(tU_\epsilon) < \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}.$$

The result follows from Proposition 5.4. □

We now consider the case $Q_M>2^{\frac{2}{N-2}}Q_m$.

Theorem 5.3. *Let $Q_M>2^{\frac{2}{N-2}}Q_m$. Suppose that*

$$|\Omega|^{\frac{N}{2}}\left(\frac{Q_M}{\int_\Omega Q(x)\,dx}\right)^{\frac{N-2}{2}} < S^{\frac{N}{2}}, \tag{5.6}$$

then problem (5.1) *admits a solution for every $\lambda>0$.*

Proof. To apply Proposition 5.4 we show that

$$\max_{t\geq 0} J_{\lambda,q}(t) < \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}. \tag{5.7}$$

We have

$$J_{\lambda,q}(t) \leq \frac{t^2}{2}|\Omega| - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)\,dx := f(t)$$

and

$$\max_{t\geq 0} f(t) = \frac{1}{N}\frac{|\Omega|^{\frac{2^*}{2^*-2}}}{\left(\int_\Omega Q(x)\,dx\right)^{\frac{2}{2^*-2}}}.$$

Assumption (5.6) guarantees (5.7). □

One can also use the estimate

$$J_{\lambda,q}(t) \leq \frac{t^2}{2}|\Omega| - \frac{t^q}{q}\lambda|\partial\Omega|$$

to derive (5.7). However, this leads to some restriction of values of a parameter λ.

6. Generalizations

In this section we consider the problem

$$\begin{cases} -\Delta u & = Q(x)u^{2^*-1} \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} & = \lambda u^{q-1} \text{ on } \partial\Omega,\ u > 0 \text{ on } \Omega. \end{cases} \tag{6.1}$$

A function $u \in H^1(\Omega)$ is a solution of problem (6.1) if

$$\int_\Omega \nabla u \nabla \phi\,dx = \int_\Omega Q(x)u^{2^*-1}\phi\,dx + \lambda\int_{\partial\Omega} u^{q-1}\phi\,dS_x \tag{6.2}$$

for every $\phi \in H^1(\Omega)$. First, we observe that if $Q(x) > 0$ on Ω, then problem (6.1) does not have a solution. Assuming that u is a solution of (6.1) and testing (6.2) with $\phi = 1$, we obtain

$$\int_\Omega Q(x)U^{2^*-1}\,dx + \lambda\int_{\partial\Omega} u^{q-1}\,dS_x = 0.$$

Since $u > 0$ on Ω, we obtain a contradiction. Therefore, we assume

(Q) $\int_\Omega Q(x)\,dx < 0$ and $Q^+(x) \neq 0,\ Q^-(x) \neq 0$.

By J_λ we denote the variational functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\,dx - \frac{\lambda}{q}\int_{\partial\Omega}|u|^q\,dS_x,$$

which is of class C^1 on $H^1(\Omega)$. We use a decomposition of $H^1(\Omega)$

$$H^1(\Omega) = V \oplus \mathbb{R},$$

where

$$V = \{v \in H^1(\Omega) \mid \int_\Omega v\,dx = 0\}.$$

Using this decomposition, we define an equivalent norm on $H^1(\Omega)$ by

$$\|u\|_V^2 = \int_\Omega |\nabla u|^2\,dx + t^2, \quad \text{if } u = v + t,\ v \in V,\ t \in \mathbb{R}.$$

To show that J_λ has a mountain-pass geometry, we need the following qualitative statement (see [5])

(S) there exists a number $\eta > 0$ such that for each $t \in \mathbb{R}$ and $v \in V$ the inequality

$$\left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{1}{2}} \leq \eta|t|$$

implies

$$\int_\Omega Q(x)|t + v(x)|^{2^*}\,dx \leq \frac{|t|^{2^*}}{2}\int_\Omega Q(x)\,dx.$$

This follows from the continuity of the embedding of V into $L^{2^*}(\Omega)$.

Proposition 6.1. *Let $1 < q < 2^{**}$ and suppose that $(\mathbf{Q})$ holds. Then there exist $\beta > 0$, $\rho > 0$ and $\lambda^* > 0$, such that for every $0 < \lambda < \lambda^*$, we have*

$$J_\lambda(u) \geq \beta \ \text{ for } \|u\| = \rho \ \text{ and } \inf_{\|u\|\leq\rho} J_\lambda(u) < 0.$$

Proof. We distinguish two cases: (i) $\|\nabla v\|_2 \leq \eta|t|$ and (ii) $\|\nabla v\|_2 > \eta|t|$, where $\eta > 0$ is a constant from $(\mathbf{S})$. If $\|\nabla v\|_2 \leq \eta|t|$ and $\|\nabla v\|_2^2 + t^2 = \rho^2$, then $t^2 \geq \frac{\rho^2}{1+\eta^2}$. By $(\mathbf{S})$ we obtain

$$\int_\Omega Q(x)|t + v(x)|^{2^*}\,dx \leq \frac{|t|^{2^*}}{2}\int_\Omega Q(x)\,dx = -|t|^{2^*}\alpha,$$

where $\alpha = -\frac{1}{2}\int_\Omega Q(x)\,dx > 0$. Hence we have the following estimate of J_λ from below

$$J_\lambda(u) \geq \frac{\alpha}{2^*}|t|^{2^*} - \frac{\lambda}{q}\int_\Omega |u|^q\,dS_x \geq \frac{\alpha\rho^{2^*}}{2^*\left(1+\eta^2\right)^{\frac{2^*}{2}}} - \frac{\lambda}{q}\int_{\partial\Omega}|u|^q\,dS_x. \tag{6.3}$$

In case (ii) we have

$$\|u\|_V \leq \|\nabla v\|_2\Big(1+\frac{1}{\eta^2}\Big)^{\frac{1}{2}}.$$

By the Sobolev inequality we have

$$|\int_\Omega Q(x)|u|^{2^*}\,dx| \leq C_1\|u\|_V^{2^*} \leq C_1\|\nabla v\|_2^{2^*}\Big(1+\frac{1}{\eta^2}\Big)^{\frac{2^*}{2}}$$

for some constant $C_1 > 0$. Hence

$$J_\lambda(u) \geq \frac{1}{2}\|\nabla v\|_2^2 - \frac{C_1}{2^*}\|\nabla v\|_2^{2^*}\Big(1+\frac{1}{\eta^2}\Big)^{\frac{2^*}{2}} - \frac{\lambda}{q}\int_{\partial\Omega}|u|^q\,dS_x.$$

Taking $\|\nabla v\|_2 \leq \rho$ small enough, we derive from this inequality the following estimate from below:

$$J_\lambda(u) \geq \frac{1}{4}\|\nabla v\|_2^2 - \frac{\lambda}{q}\int_{\partial\Omega}|u|^q\,dS_x.$$

If $\|u\|_V = \rho$, then $\rho \leq \|\nabla v\|_2\frac{(1+\eta^2)^{\frac{1}{2}}}{\eta}$. Therefore,

$$J_\lambda(u) \geq \frac{\eta^2\rho^2}{4(1+\eta^2)} - \frac{\lambda}{q}\int_{\partial\Omega}|u|\,dS_x. \tag{6.4}$$

From (6.3) and (6.4) we derive

$$J_\lambda(u) \geq \min\Big(\frac{\eta^2\rho^2}{4(1+\eta^2)}, \frac{\alpha\rho^{2^*}}{2^*(1+\eta^2)^{\frac{2^*}{2}}}\Big) - \frac{\lambda}{q}\int_{\partial\Omega}|u|^q\,dS_x.$$

Applying the Sobolev trace embedding theorem, we get

$$J_\lambda(u) \geq \min\Big(\frac{\eta^2\rho^2}{4(1+\eta^2)}, \frac{\alpha\rho^{2^*}}{2^*(1+\eta^2)^{\frac{2^*}{2}}}\Big) - \lambda C_2\|u\|_V^q$$

for some constant $C_2 > 0$. Taking $\lambda^* > 0$ small enough the first assertion follows. Since $1 < q < 2^{**} < 2^*$ we have

$$J_\lambda(u) = -\frac{|t|^{2^*}}{2^*}\int_\Omega Q(x)\,dx - \frac{\lambda}{q}|t|^q|\partial\Omega| < 0$$

for t sufficiently small. Hence $\inf_{\|u\|\leq\rho} J_\lambda(u) < 0$. □

Repeating the proof of Theorem 3.1 we get

Theorem 6.1. *Let $1 < q < 2^{**}$. Then for every $0 < \lambda < \lambda^*$ there exists a solution u of problem (6.1) satisfying $J_\lambda(u) < 0$.*

Let us denote by u_λ a solution of problem (6.1) from Theorem 6.1.

Proposition 6.2. *Let $1 < q < 2$. Suppose that $Q(x) > 0$ somewhere on $\partial\Omega$. If $u = 0$ and $u = u_\lambda$ are the unique critical points of J_λ, then the $(PS)_c$ condition holds for*

$$c < \bar{c} = J_\lambda(u_\lambda) + \min\left(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}\right).$$

The proof is analogous to that of Proposition 4.1 and is omitted.

From this we can derive the first existence result for problem (6.1).

Theorem 6.2. *Let $N \geq 5$, $1 < q < 2$ and $Q_M \leq 2^{\frac{2}{N-2}} Q_m$. Suppose that $Q_m = Q(0)$, $0 \in \partial\Omega$, $H(0) > 0$ and that assumption* (4.11) *holds. Then problem* (6.1) *admits a solution for every $0 < \lambda < \lambda^*$.*

We now consider the case $Q_M > 2^{\frac{2}{N-2}} Q_m$. In this case

$$\bar{c} = J_\lambda(u_\lambda) + \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}.$$

Let $Q_M = Q(x_M)$, $x_M \in \Omega$.

Theorem 6.3. *Let $1 < q < 2$ and $Q_M > 2^{\frac{2}{N-2}} Q_m$. Suppose that*

$$|Q(x) - Q(x_M)| = o\big(|x - x_M|^{\frac{N-2}{2}}\big) \tag{6.5}$$

for x near x_M. Then problem (6.1) *admits a second solution for every $0 < \lambda < \lambda^*$.*

Proof. A second solution will be obtained through the mountain-pass principle. Let $\phi \in C^1(\mathbb{R}^N)$ be such that $\phi(x) = 1$ on $B(x_M, \frac{r}{2})$, $\phi(x) = 0$ on $\mathbb{R}^N - B(x_M, r)$ and $0 \leq \phi(x) \leq 1$, $B(x_M, r) \subset \{x \in \Omega \,|\, Q(x) > 0\}$. We set $W_\epsilon = \phi U_{\epsilon, x_M}$. It is sufficient to show that

$$\max_{t \geq 0} J_\lambda(u_\lambda + tW_\epsilon) \leq J_\lambda(u_\lambda) + \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}. \tag{6.6}$$

We have, as in the proof of Proposition 4.2,

$$\begin{aligned}
J_\lambda(u_\lambda + tW_\epsilon) &\leq J_\lambda(u_\lambda) + \frac{t^2}{2}\int_\Omega |\nabla W_\epsilon|^2\,dx + \frac{1}{2^*}\int_\Omega Q^+(x)u_\lambda^{2^*}\,dx \\
&- \frac{1}{2^*}\int_\Omega Q^+(x)\big(u_\lambda + tW_\epsilon\big)^{2^*}\,dx + t\int_\Omega Q^+(x)u_\lambda^{2^*-1}W_\epsilon\,dx \\
&\leq J_\lambda(u_\lambda) + \frac{t^2}{2}\int_\Omega |\nabla W_\epsilon|^2\,dx - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)W_\epsilon^{2^*}\,dx \\
&- t^{2^*-1}\int_\Omega Q(x)W_\epsilon^{2^*-1}u_\lambda\,dx \\
&+ Ct^{\frac{N+1}{N-2}}\int_\Omega Q(x)u_\lambda^{\frac{N-1}{N-2}}W_\epsilon^{\frac{N+1}{N-2}}\,dx \\
&= J_\lambda(u_\lambda) + \Psi_\epsilon(t).
\end{aligned}$$

Let $\Psi_\epsilon(t_\epsilon) = \max_{t\geq 0}\Psi_\epsilon(t)$. Repeating the proof of Proposition 4.2 we can show that there exist constants $0 < T_1 < T_2$ such that $T_1 \leq t_\epsilon \leq T_2$. Since $u_\lambda > 0$ on Ω, there exists a constant $\alpha > 0$ such that

$$\int_\Omega Q(x)u_\lambda W_\epsilon^{2^*-1}\,dx \geq \alpha\epsilon^{\frac{N-2}{2}}.$$

Hence

$$\begin{aligned}
J_\lambda(u_\lambda + tW_\epsilon) &\leq J_\lambda(u_\lambda) + \frac{\Big(\int_\Omega |\nabla W_\epsilon|^2\,dx\Big)^{\frac{N}{2}}}{N\Big(\int_\Omega Q(x)W_\epsilon^{2^*}\,dx\Big)^{\frac{N-2}{2}}} + CT_2^{\frac{N-1}{2}}\epsilon^{\frac{N-1}{2}} \\
&- \alpha T_1^{2^*-1}\epsilon^{\frac{N-2}{2}}.
\end{aligned}$$

Assumption (6.5) yields the following expansion of $\int_\Omega Q(x)W_\epsilon^{2^*}\,dx$:

$$\int_\Omega Q(x)W_\epsilon^{2^*}\,dx = Q_M\int_\Omega W_\epsilon^{2^*}\,dx + o\big(\epsilon^{\frac{N-2}{2}}\big).$$

We also have

$$\int_\Omega |\nabla W_\epsilon|^2\,dx = S^{\frac{N}{2}} + O\big(\epsilon^{N-2}\big) \quad\text{and}\quad \int_\Omega W_\epsilon^{2^*}\,dx = S^{\frac{N}{2}} + O\big(\epsilon^{N}\big).$$

Hence

$$\max_{t\geq 0} J_\lambda(u_\lambda + tW_\epsilon) < J_\lambda(u_\lambda) + \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}$$

for small $\epsilon > 0$. □

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