Annals of the University of Bucharest (mathematical series)
2 (LX) (2011), 3–17

Free boundary seepage from open earthen channels

Adrian Carabineanu

Communicated by Olivian Simionescu-Panait

Abstract - We present in this paper a new inverse method for the study of the seepage from symmetrical earthen open channels. We map the half-strip from the complex potential domain onto the unit half-disk from an auxiliary ζ - plane. We introduce Levi-Civitá's function whose imaginary part vanishes on the diameter $[-1, 1]$ of the half-disk by virtue of the boundary conditions imposed on the free lines. Then we give integral representations of the complex potential, velocity field, free phreatic lines and contour of the channel by means of σ, the real part of Levi-Civitá's function on the unit half-circle. For various values of the Fourier coefficients of σ we calculate numerically the contour of the channel, the phreatic lines, the seepage loss, the velocity field, the streamlines and the equipotential lines.

Key words and phrases : Darcy law, conformal mapping, Levi-Civitá's function, phreatic lines, seepage loss.

Mathematics Subject Classification (2000) **:** 76S05, 76B07, 76M40.

1. Introduction

The free surface seepage may be encountered in many engineering problems involving the flow of water through a porous medium, such as the permeable dam problem [9], [10], [12], irrigation and drainage problems [3], [4] or the seepage from earthen channels [5], [11], [14], [15], [18], [21], etc. Mathematically, these problems reduce to boundary values problems for partial differential equations of elliptic type in domains with partially unknown boundaries that are found using specified boundary conditions. One of the most powerful mathematical techniques utilized for studying the free boundary seepage problems is the theory of variational inequalities (see for example Oden and Kikuchi [18], Chipot [8] and Chipot and Lyaghfouri [9], [10]). Using this technique and the finite element method, Oden and Kikuchi [18] solved numerically some problems involving fluid seepage with free boundaries in porous media.

In the case of homogeneous isotropic porous media, employing the theory of complex holomorphic functions or the theory of real harmonic functions, Carabineanu [3], Carabineanu and Honţuş [4], France et all. [12] and others utilized the boundary element method for solving numerically some $2d$

problems concerning the irrigation, drainage or seepage through permeable dams.

Besides the numerical methods, the analytical ones are necessary not only to substantiate and test numerical algorithms but also to gain a deeper understanding of the underlying physics. A valuable survey on the analytical studies of the problems arising in conection with the free boundary seepage is the paper of Ilyinsky, Kacimov and Yakimov [15].

In the case of the $2d$ seepage from soil channels into a homogeneous isotropic porous medium, one knows the complex potential domain (a half-strip), a part of the boundary of the physical flow domain (the contour of the channel), a part of the boundary of the Zhukovskii complex domain (the free lines) and a part of the boundary of the complex velocity domain (the free lines). This fact suggests the elaboration of various variants of the inverse method. The inverse method does not solve the *direct seepage problem: given the contour of the channel, calculate the corresponding seepage loss*, but Kacimov's comparison theorem [16], which states that *for any arbitrary channel, the seepage loss is bounded from below and above by the seepage discharges from an arbitrary inscribed channel and an arbitrary comprising channel respectively*, is a valuable tool for studying the direct problem by means of the inverse method. To this aim, it is important to have a great number of contours of channel obtained by means of the inverse methods. We shall review therefore some papers where various alternatives of the inverse method for the seepage problem from soil channels have been employed: Kozeny (see [20], [13]) studied the seepage from a curved channel using Zhukovskii's function and found that the resultant channel has trochoid form. In [1], Anakhaev obtained a solution for curvilinear watercourses by representing the watercourse profiles in the Zhukovskii plane by means of the equation of a family of lemniscates. Other types of watercourses with different relative widths where studied by Anakhaev in [2]. For the particular case of a circular base of the watercourse profile, the solution of Anakhaev coincides with the known exact solutions derived by Vedernikov [22] and Pavlovskii [19]. Chahar utilized in [6] the inverse method to obtain an exact solution for seepage from a curved channel whose boundary maps along a circle onto the complex velocity plane. The channel shape is an approximate semiellipse with the top width as the major axis and twice the water depth as the minor axis. In a subsequent paper dedicated to the same class of curvilinear bottomed channels Chahar [7] discusses the optimal section properties from the least area and minimum seepage loss points of view. Kacimov and Obsonov [17] used the inverse method to find the shape of a soil channel of constant hydraulic gradient. In [17] and in [14] the authors utilized an inverse method where the shape of the unknown channel is searched as part of the solution.

In the present paper, we present a new variant of the complex velocity-complex potential pair, based on Levi-Civitá's function. We consider the conformal mapping $f(\zeta)$ of the unit half-disk onto the half-strip from the complex potential plane. Instead of searching the conformal mapping of the half-strip from the complex potential plane onto the complex velocity domain we shall look for Levi-Civitá's function $\omega(\zeta)$. The radii $(-1,0)$ and $(0,1)$ of the unit half-disk correspond through the conformal mapping $z(\zeta)$ to the free (phreatic) lines from the flow domain. On these radii, the imaginary part of $\omega(\zeta)$ vanishes by virtue of the conditions imposed on the free lines. According to Schwarz's principle of symmetry, we may extend the domain of definition of $\omega(\zeta)$ to the whole unit disk and we may employ Schwarz-Villat's integral formula for representing $\omega(\zeta)$ by means of $\sigma(s) = \mathit{Real}\, \omega(\exp(is)),\ s \in [0,\pi]$.

At the first glance, because of the integral representations that we have to employ, the inverse method proposed herein seems to be more complicated than the classical inverse method involving the conformal mapping of the half-strip onto the complex velocity domain. However, in the end, we have to give only the expression of the function $\sigma(s)$ (in fact we shall give the coefficients of the Fourier series of σ) in order to construct the channel profile and solve the corresponding free boundary seepage problem. In Section 6, we present some calculated channel profiles and the corresponding phreatic lines, stream lines, equipotential lines and seepage losses. The integrals occurring in the corresponding integral representations are calculated numerically. In fact, we have conceived a Matlab code. The input data consist of the coefficients of the Fourier series of $\sigma(s)$. The output consists of the seepage loss and phreatic lines, streamlines and equipotential lines calculated in the nodes of a mesh from the flow domain. In some particular cases we compare the numerical results with analytical ones and we notice a very good agreement.

2. The free boundary value problem

From the equation of continuity

$$\operatorname{div} \mathbf{v} = 0 \tag{2.1}$$

and Darcy's law for a homogeneous isotropic poros medium

$$\mathbf{v} = \operatorname{grad} \varphi,\ \varphi = -k\left(\frac{p}{\rho g} + y\right) + \mathit{const.}, \tag{2.2}$$

we deduce that

$$\Delta\varphi = 0. \tag{2.3}$$

Here φ is the potential of the velocity, $\mathbf{v} = (u, v)$ is the velocity, p - the pressure and ρ - the density of the fluid, k is the constant filtration coefficient (hydraulic conductivity), g is the gravity constant and (x, y) are the cartesian coordinates.

Let ψ (the stream function) be the harmonic conjugate of φ. For $z = x+iy$ the analytic function $f(z) = \varphi(x,y)+i\psi(x,y)$ is the complex potential and

$$\frac{df}{dz} = \frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} = w,. \tag{2.4}$$

where

$$w = u - iv \tag{2.5}$$

is the complex velocity

Now we are going to establish the boundary conditions.

We consider a soil channel whose profile is a curve which has the following equation:

$$y = y(x)\,,\ x \in [-L, L]\,,\ y(L) = y(-L) = 0. \tag{2.6}$$

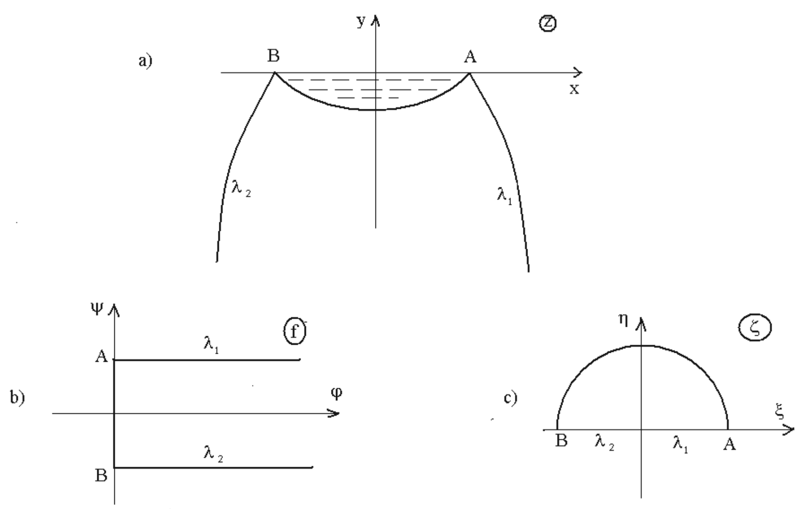

Figure 1: a) Flow domain in the porous medium. b) Half-strip in the plane of the complex potential. c) Half-disk

Let $y = 0$ be the level of the water in the channel (figure 1.a)). Assuming that there is no lining of the bottom AB of the channel, the pressure on AB is

$$p = p_{atm} - \rho g y,$$

(p_{atm} is the atmospheric pressure), whence we deduce that

$$\varphi\,|_{AB} = 0. \tag{2.7}$$

On AB the tangential velocity $\dfrac{\partial\varphi}{\partial s}$ vanishes, hence we have

$$\arg w(z)\,|_{AB} = \arg(u - iv) = -\arctan\frac{dy}{dx} + \frac{\pi}{2}. \tag{2.8}$$

The free boundaries (phreatic lines) λ_1 and λ_2 are streamlines, whence

$$\psi\,|_{\lambda_1}=\frac{Q}{2},\ \psi\,|_{\lambda_2}=-\frac{Q}{2}, \tag{2.9}$$

where Q is the seepage loss (discharge). We study the seepage flow without capilarity. Hence on the free phreatic lines the pressure has the constant value p_{atm}. We have therefore

$$\varphi+ky\,|_{\lambda_1\cup\lambda_2}=0. \tag{2.10}$$

Derivating along the tangential direction we get

$$\frac{\partial\varphi}{\partial s}+k\frac{\partial y}{\partial s}\,|_{\lambda_1\cup\lambda_2}=0,\ \text{whence}\ u^2+v^2+kv\,|_{\lambda_1\cup\lambda_2}=0. \tag{2.11}$$

3. Levi-Civitá's function

From (2.7) and (2.9) it follows that the image of the domain of motion in the plane of the complex potential is a half-strip (figure 1.b)).

The function

$$f=-\frac{Q}{\pi}\ln\zeta+\frac{Qi}{2},\ \zeta=\xi+i\eta,\ \ln(-1+0i)=\pi i,\ \ln(-1-0i)=-\pi i, \tag{3.1}$$

is the conformal mapping of the unit half-disk from the ζ - plane (figure 1.c)) onto the half-strip from the f - plane. From (2.4) and (3.1) we deduce that f, z and w may be regarded as functions of ζ ($z(\zeta)$ is the conformal mapping of the unit half-disk from the ζ - plane onto the flow domain from the z - plane). The free lines λ_1 and λ_2 represent the image of the real diameter $\zeta=\xi+i\eta,\ \eta=0,\ \xi\in[-1,0)\cup(0,1]$ by the conformal mapping $z(\zeta)$ and the contour of the channel is the imagine of the half-circle $\zeta=\exp(is),\ s\in[0,\pi]$ by the same mapping.

We shall continue by introducing the auxiliary function $w^*(\zeta)$ by means of the relation

$$w^*(\zeta)=w(\zeta)-\frac{ik}{2}=u-i\left(v+\frac{k}{2}\right). \tag{3.2}$$

From (2.11) and (3.2) it results

$$V^*(\xi)=|w^*(\xi)|=\frac{k}{2},\ \xi\in[-1,0)\cup(0,1]. \tag{3.3}$$

In the sequel we introduce Levi-Civitá's $\omega(\zeta)=\sigma(\xi,\eta)+i\tau(\xi,\eta)$ by means of the relation

$$w^*(\zeta)=\frac{k}{2}\exp(-i\omega(\zeta)). \tag{3.4}$$

Since $w^*=V^*\exp(i\arg w^*)$, from (3.4), we deduce that

$$\sigma=-\arg w^*,\ \tau=\ln\frac{2V^*}{k}. \tag{3.5}$$

4. Integral representations and conformal mappings

From (3.3) and (3.5) it follows that

$$\tau(\xi,0)=0,\ \xi\in[-1,0)\cup(0,1]. \tag{4.1}$$

According to Schwarz's principle of symmetry, the function $\omega(\zeta)$ can be extended to the whole unit disk by means of the relation

$$\omega(\zeta)=\overline{\omega}\left(\overline{\zeta}\right),\ \text{whence } \sigma(\xi,\eta)=\sigma(\xi,-\eta),\ \tau(\xi,\eta)=-\tau(\xi,-\eta). \tag{4.2}$$

Applying Schwarz-Villat's formula and (4.2) we get the integral representation

$$\omega(\zeta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma\left(s'\right)\frac{\exp\left(is'\right)+\zeta}{\exp\left(is'\right)-\zeta}ds'=\frac{1}{\pi}\int_{0}^{\pi}\sigma\left(s'\right)\frac{1-\zeta^2}{1-2\zeta\cos s'+\zeta^2}ds', \tag{4.3}$$

where we denoted $\sigma(s')=\sigma(\cos s',\sin s')$. We denote $\tau(s')=\tau(\cos s',\sin s')$.

If the function $\sigma(s)$ satisfies a Hölder condition, employing Plemelj's formula we obtain the following relation on the boundary of the unit disk

$$\omega(\exp(is))=\sigma(s)-\frac{i}{\pi}{\int_{0}^{\pi}}'\sigma\left(s'\right)\frac{\sin s}{\cos s-\cos s'}ds',$$

whence, separating the imaginary parts, we get

$$\tau(s)=-\frac{1}{\pi}{\int_{0}^{\pi}}'\sigma\left(s'\right)\frac{\sin s}{\cos s-\cos s'}ds'. \tag{4.4}$$

The prime "′" indicates the Cauchy principal value of the singular integral. From the relation

$$\frac{df}{dz}=w(\zeta)=w^*(\zeta)+\frac{ik}{2}, \tag{4.5}$$

it follows

$$dz=\frac{df}{w^*(\zeta)+\dfrac{ik}{2}} \tag{4.6}$$

Taking into account (3.1) and (3.4) we obtain

$$dz=\frac{-2Q}{k\pi\left(i+\exp\left(-i\omega(\zeta)\right)\right)}\frac{d\zeta}{\zeta}, \tag{4.7}$$

whence it results the following integral representation of the conformal mapping $z(\zeta)$:

$$z(\zeta)=z(\zeta_0)-\int_{\zeta_0}^{\zeta}\frac{2Q}{k\pi\left(i+\exp\left(-i\omega(\zeta)\right)\right)}\frac{d\zeta}{\zeta}. \tag{4.8}$$

Considering in (4.7) $\zeta = \exp(is)$ we get

$$dx + idy = \frac{-2Qids}{k\pi\left(i + \exp\left(-i\sigma\left(s\right) + \tau\left(s\right)\right)\right)} \tag{4.9}$$

on the profile of the channel. Separating in (4.9) the real parts and the imaginary ones, we obtain

$$\frac{dx}{ds} = -\frac{2Q}{k\pi}\frac{1 - \sin\sigma\left(s\right)\exp\left(\tau\left(s\right)\right)}{1 - 2\sin\sigma\left(s\right)\exp\left(\tau\left(s\right)\right) + \exp\left(2\tau\left(s\right)\right)}, \tag{4.10}$$

$$\frac{dy}{ds} = -\frac{2Q}{k\pi}\frac{\cos\sigma\left(s\right)\exp\left(\tau\left(s\right)\right)}{1 - 2\sin\sigma\left(s\right)\exp\left(\tau\left(s\right)\right) + \exp\left(2\tau\left(s\right)\right)}, \tag{4.11}$$

whence it follows

$$x\left(s\right) = L - \frac{2Q}{k\pi}\int_0^s \frac{1 - \sin\sigma\left(s\right)\exp\left(\tau\left(s\right)\right)}{1 - 2\sin\sigma\left(s\right)\exp\left(\tau\left(s\right)\right) + \exp\left(2\tau\left(s\right)\right)}ds, \tag{4.12}$$

$$y\left(s\right) = -\frac{2Q}{k\pi}\int_0^s \frac{\cos\sigma\left(s\right)\exp\left(\tau\left(s\right)\right)}{1 - 2\sin\sigma\left(s\right)\exp\left(\tau\left(s\right)\right) + \exp\left(2\tau\left(s\right)\right)}ds. \tag{4.13}$$

5. Some properties of Levi-Civitá's function in the case of symmetrical channels

The functions $\sigma(s)$ and $\tau(s)$ are not known, but we know the angle of the velocity with the Ox -axis on the profile of the channel:

$$\theta\left(x\right) = -\arg w\left(x + iy\right).$$

Denoting

$$\theta\left(s\right) = \theta\left(x\left(s\right)\right),\ V\left(s\right) = \left|w\left(\exp\left(is\right)\right)\right|,$$

we obtain, from (3.2), the relation

$$V\left(s\right)\exp\left(-i\theta\left(s\right)\right) = V^*\left(s\right)\exp\left(-i\sigma\left(s\right)\right) + \frac{ik}{2}. \tag{5.1}$$

Separating the real and imaginary parts we get

$$V\cos\theta\left(s\right) = V^*\cos\sigma\left(s\right),\ V\sin\theta\left(s\right) = V^*\sin\sigma\left(s\right) - \frac{k}{2}, \tag{5.2}$$

whence, taking into account (3.5), it follows

$$\sin\left(\sigma\left(s\right) - \theta\left(s\right)\right) = \frac{\cos\theta\left(s\right)}{\exp\left(\tau\left(s\right)\right)}. \tag{5.3}$$

Instead of relation (5.3) we may consider one of the following relations:

$$\sigma\left(s\right) = \theta\left(s\right) + \arcsin\frac{\cos\theta\left(s\right)}{\exp\left(\tau\left(s\right)\right)}, \tag{5.4}$$

$$\sigma(s) = \pi + \theta(s) - \arcsin \frac{\cos\theta(s)}{\exp(\tau(s))}. \tag{5.5}$$

Now we shall assume that the profile of the channel is symmetric i.e.

$$y(x) = y(-x). \tag{5.6}$$

Due to the symmetry of the channel we have the relations

$$\theta(\pi - s) = -\pi - \theta(s), \tag{5.7}$$

$$V(\pi - s) = V(s). \tag{5.8}$$

From (3.2), (3.5), (5.7) and (5.8) it follows

$$V^*(s) = V^*(\pi - s),\ \tau(s) = \tau^*(\pi - s). \tag{5.9}$$

Let us assume that for the channel we have in view, the relation (5.4) is valid. In this case, from (5.7) and (5.9), it results

$$\sigma(\pi - s) = -\pi - \theta(s) - \arcsin \frac{\cos\theta(s)}{\exp(\tau(s))} = -\pi - \sigma(s). \tag{5.10}$$

From (5.10) it results that

$$\omega(0) = \frac{1}{\pi}\int_0^\pi \sigma(s)\,ds = -\frac{\pi}{2}. \tag{5.11}$$

Therefore, at infinity, in the flow domain

$$w^* = \frac{1}{2}k\exp(-i\omega(0)) = \frac{ik}{2} \Rightarrow w = ik. \tag{5.12}$$

This happens when there is a drain at infinity.

Let us further assume that the relation (5.3) is equivalent to the relation (5.5). From (5.5), (5.7) and (5.9) it follows that

$$\sigma(\pi - s) = -\theta(s) + \arcsin \frac{\cos\theta(s)}{\exp(\tau(s))} = \pi - \sigma(s), \tag{5.13}$$

whence

$$\omega(0) = \frac{1}{\pi}\int_0^\pi \sigma(s)\,ds = \frac{\pi}{2}. \tag{5.14}$$

Therefore, at infinity, in the flow domain

$$w^* = \frac{1}{2}k\exp(-i\omega(0)) = -\frac{ik}{2}\ \text{i.e.}\ w = 0. \tag{5.15}$$

This happens when at infinity there is an impermeable layer.

6. An inverse method for the study of the water seepage from symmetrical channels

6.1. Drain at infinity

Let

$$\lim_{s\to\pi/2, s\geq\pi/2} \sigma(s) = \mu\pi. \tag{6.1}$$

From (5.10) it follows that

$$\sigma(s) = \mu\pi + \sigma_0(s), \tag{6.2}$$

$$\sigma(\pi - s) = -\pi - \mu\pi + \sigma_0(\pi - s), \ s \in \left(0, \frac{\pi}{2}\right), \tag{6.3}$$

where σ_0 is a continuous function such that

$$\sigma_0\left(\frac{\pi}{2}\right) = 0, \ \sigma_0(\pi - s) = -\sigma_0(s), \ s \in \left(0, \frac{\pi}{2}\right). \tag{6.4}$$

According to (4.2), we may extend the domain of definition of σ_0 to $[0, 2\pi]$ by putting

$$\sigma_0(2\pi - s) = \sigma_0(s), \ s \in (0, \pi). \tag{6.5}$$

From (6.4) and (6.5) we deduce that the Fourier series of $\sigma_0(s)$ is

$$\sigma_0(s) = \sum_{n=0}^{\infty} a_{2n+1}\cos(2n+1)s, \ a_{2n+1} \in \mathbf{R}, \ n = 0, 1, ... \tag{6.6}$$

From (4.3), (6.3) and (6.6) we get

$$\omega(\zeta) = (1 + 2\mu)\, i \ln\frac{\zeta + i}{\zeta - i} + 2\mu\pi + \frac{\pi}{2} + \sum_{n=0}^{\infty} a_{2n+1}\zeta^{2n+1}, \tag{6.7}$$

$$\tau(s) = \ln\tan^{1+2\mu}\left(\frac{s}{2} + \frac{\pi}{4}\right) + \tau_0(s), \tag{6.8}$$

with

$$\tau_0(s) = \sum_{n=0}^{\infty} a_{2n+1}\sin(2n+1)s.$$

Introducing (6.3), (6.6) and (6.8) in (4.12) and (4.13), one obtains the parametric equations of the profile of the channel. The integrals may be calculated numerically (we may employ for example the trapeziums formula). From (4.8) and (6.7) we obtain the conformal mapping $z(\zeta)$:

$$z(\zeta) = z(\zeta_0) - \frac{2Q}{k\pi i}\int_{\zeta_0}^{\zeta} \frac{1}{1 - \left(\dfrac{\zeta + i}{\zeta - i}\right)^{1+2\mu} \exp\left(-2\mu\pi i - i\displaystyle\sum_{n=0}^{\infty} a_{2n+1}\zeta^{2n+1}\right)}\frac{d\zeta}{\zeta}. \tag{6.9}$$

If $\zeta = r\exp(is)$, $0 < r < 1$, we choose $\zeta_0 = \exp(is)$ and $z(\zeta_0) = x(s) + iy(s)$. The path of integration is the segment $[\zeta_0, \zeta]$ and the integral may be calculated numerically using trapeziums formula or another approximate integration formula.

The parametric equation of the phreatic lines λ_1 and λ_2 are

$$z(\xi) = L - \frac{2Q}{k\pi i}\int_1^{\xi} \frac{1}{1 - \left(\frac{\xi+i}{\xi-i}\right)^{1+2\mu}\exp\left(-2\mu\pi i - i\sum_{n=0}^{\infty} a_{2n+1}\xi^{2n+1}\right)}\frac{d\xi}{\xi},$$

$$\xi \in (0,1]. \tag{6.10}$$

$$z(\xi) = -L - \frac{2Q}{k\pi i}\int_{-1}^{\xi} \frac{1}{1 - \left(\frac{\xi+i}{\xi-i}\right)^{1+2\mu}\exp\left(-2\mu\pi i - i\sum_{n=0}^{\infty} a_{2n+1}\xi^{2n+1}\right)}\frac{d\xi}{\xi},$$

$$\xi \in [-1,0). \tag{6.11}$$

From (3.2), (3.4) and (6.7) we obtain the complex velocity in the points $z(\zeta)$:

$$w(z(\zeta)) = w(\zeta) = \frac{ik}{2}\left[1 + \left(\frac{\zeta+i}{\zeta-i}\right)^{1+2\mu}\exp\left(-2\mu\pi i - i\sum_{n=0}^{\infty} a_{2n+1}\xi^{2n+1}\right)\right]. \tag{6.12}$$

Imposing $x(\pi) = -L$ in (4.12), we get the seepage loss

$$Q = \frac{kL\pi}{2\int_0^{\pi/2} \frac{1 - \sin\sigma(s)\exp(\tau(s))}{1 - 2\sin\sigma(s)\exp(\tau(s)) + \exp(2\tau(s))}ds}. \tag{6.13}$$

For $\sigma(s) = 0$, $s \in \left[0, \frac{\pi}{2}\right)$ i.e. $\mu = 0$, $a_{2n+1} = 0, n = 0, 1, ...,$ we obtain the analytical formulas:

$$\tau(s) = \ln\tan\left(\frac{s}{2} + \frac{\pi}{4}\right), \tag{6.14}$$

$$x(s) = L - \frac{2L}{\pi - 2}(s + \cos s - 1), \tag{6.15}$$

$$y(s) = -\frac{2L}{\pi - 2}\sin s, \tag{6.16}$$

$$\theta(s) = \frac{s}{2} - \frac{\pi}{4}, \tag{6.17}$$

$$Q = \frac{2kL\pi}{\pi - 2}. \tag{6.18}$$

Equations (6.15) and (6.16) are the parametric equations of an arc of cycloid with an angular point.

From (3.2), (3.4), (4.3) and (5.10) we obtain Levi-Civitá' s function

$$\omega(\zeta) = i \ln \frac{\zeta + i}{\zeta - i} + \frac{\pi}{2}$$

and the complex velocity

$$w(\zeta) = \frac{k}{\zeta - i}. \tag{6.19}$$

From (3.1), (4.5) and (6.19) we get the conformal mapping of the unit half-disk onto the flow domain in the porous medium

$$z(\zeta) = -\frac{2L}{\pi - 2}\left(\zeta - i \ln \zeta - \frac{\pi}{2}\right), \tag{6.20}$$

as well as the parametric equations of the free lines:

$$z(\xi) = -\frac{2L}{\pi - 2}\left(\xi - i \ln \xi - \frac{\pi}{2}\right);\ \xi \in [-1, 0) \cup (0, 1]. \tag{6.21}$$

In figure 2 we present the seepage from channels having various profiles. For the graphic representations we employ dimensionless variables:

$$x^* = \frac{x}{L}, y^* = \frac{y}{L}, u^* = \frac{u}{k}, v^* = \frac{v}{k}, Q^* = \frac{Q}{kL}, \varphi^* = \frac{\varphi}{kL}, \psi^* = \frac{\psi}{kL}.$$

We use solid lines for the contour of the channel and the equipotential lines, broken lines for the the streamlines (including the phreatic lines λ_1 and λ_2) and arrows for the velocity field. We also indicate the numerical values of the dimensionless seepage loss Q^*. In figure 2.a) we considered $\sigma(s) = \frac{\pi}{2} + \frac{\pi}{4}\cos s,\ s \in \left[0, \frac{\pi}{2}\right]$, in figure 2.b) we considered $\sigma(s) = \frac{\pi}{4} + \frac{\pi}{4}\cos s,\ s \in \left[0, \frac{\pi}{2}\right]$, in figures 2.c) and d) we considered $\sigma(s) = 0,\ s \in \left[0, \frac{\pi}{2}\right]$. We notice a very good agreement between the numerical results obtained in figure 2.c) and the analytical results obtained in figure 2.d) (by means of formulas (6.14) - (6.21) for $\sigma = 0$.

For the numerical computations we utilized in the ζ - complex plane the mesh points $\left\{ \zeta_{jl} = \frac{j \exp\left(\frac{l\pi}{m}\right)}{n}, j = 1, 2, ..., n,\ l = 0, 1, ..., m \right\}$. In the flow domain we considered the mesh points $z\left(\zeta_{jl}\right)$ obtained by means of the conformal mapping (6.9). In this paper we take $m = 20, n = 30$.

We have to mention that for some values of the Fourier coefficients of σ we may obtain results which are unacceptable from a physical point of view: self-intersecting channel profiles, self-intersecting phreatic lines, positive values of the vertical coordinates of the velocity. We find that the results are unacceptable after examinig the graphic representations given by the Matlab code.

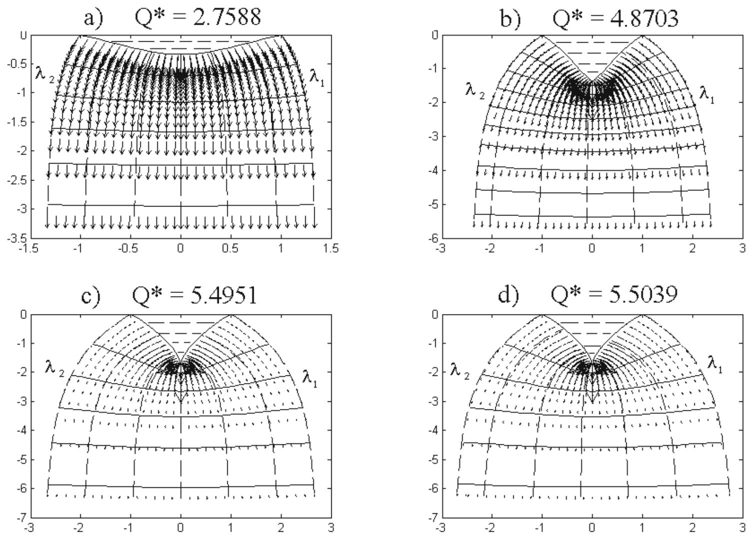

Figure 2. Seepage from various channels. Drain at infinity

6.2. Impermeable layer at infinity

Like in the previous case, performing similar calculations, we get:

$$\sigma(s) = \mu\pi + \sigma_0(s),$$

$$\sigma(\pi - s) = -\pi - \mu\pi + \sigma_0(\pi - s), \; s \in \left(0, \frac{\pi}{2}\right), \tag{6.22}$$

$$\omega(\zeta) = (-1 + 2\mu)\, i \ln \frac{\zeta + i}{\zeta - i} + 2\mu\pi - \frac{\pi}{2} + \sum_{n=0}^{\infty} a_{2n+1}\zeta^{2n+1}, \tag{6.23}$$

$$\tau(s) = \ln \tan^{2\mu-1}\left(\frac{s}{2} + \frac{\pi}{4}\right) + \tau_0(s), \tag{6.24}$$

$$z(\zeta) = z(\zeta_0) - \frac{2Q}{k\pi i}\int_{\zeta_0}^{\zeta} \frac{1}{1 + \left(\frac{\zeta + i}{\zeta - i}\right)^{-1+2\mu} \exp\left(-2\mu\pi i - i\sum_{n=0}^{\infty} a_{2n+1}\zeta^{2n+1}\right)}\frac{d\zeta}{\zeta}, \tag{6.25}$$

$$z(\xi) = L - \frac{2Q}{k\pi i}\int_{1}^{\xi} \frac{1}{1 + \left(\frac{\xi + i}{\xi - i}\right)^{-1+2\mu} \exp\left(-2\mu\pi i - i\sum_{n=0}^{\infty} a_{2n+1}\xi^{2n+1}\right)}\frac{d\xi}{\xi},$$

$$\xi \in (0,1]\,. \tag{6.26}$$

$$z(\xi) = -L - \frac{2Q}{k\pi i}\int_{-1}^{\xi} \frac{1}{1+\left(\dfrac{\xi+i}{\xi-i}\right)^{-1+2\mu}\exp\left(-2\mu\pi i - i\displaystyle\sum_{n=0}^{\infty} a_{2n+1}\xi^{2n+1}\right)}\frac{d\xi}{\xi},$$

$$\xi \in [-1,0)\,, \tag{6.27}$$

$$w(z(\zeta)) = w(\zeta) = \frac{ik}{2}\left[1-\left(\frac{\zeta+i}{\zeta-i}\right)^{-1+2\mu}\exp\left(-2\mu\pi i - i\sum_{n=0}^{\infty} a_{2n+1}\zeta^{2n+1}\right)\right]. \tag{6.28}$$

For $\sigma(s) = 0,\ s \in \left[0, \frac{\pi}{2}\right)$, we obtain

$$\tau(s) = -\ln\tan\left(\frac{s}{2}+\frac{\pi}{4}\right), \tag{6.29}$$

$$x(s) = \frac{\pi L}{\pi+2} + \frac{2L}{\pi+2}(\cos s - s)\,, \tag{6.30}$$

$$y(s) = -\frac{2L}{\pi+2}\sin s, \tag{6.31}$$

$$\theta(s) = -\frac{s}{2} - \frac{\pi}{4}, \tag{6.32}$$

$$Q = \frac{2kL\pi}{\pi+2}, \tag{6.33}$$

$$\omega(\zeta) = -i\ln\frac{\zeta+i}{\zeta-i} - \frac{\pi}{2},$$

$$w(\zeta) = \frac{ik\zeta}{\zeta+i}, \tag{6.34}$$

$$z(\zeta) = \frac{2Li}{\pi+2}\left(\ln\zeta - \frac{i}{\zeta}\right) + \frac{\pi L}{\pi+2}, \tag{6.35}$$

$$z(\xi) = \frac{2Li}{\pi+2}\left(\ln\xi - \frac{i}{\xi}\right) + \frac{\pi L}{\pi+2};\ \xi \in [-1,0)\cup(0,1]\,. \tag{6.36}$$

Equations (6.30) and (6.31) are the parametric equations of an arc of cycloid.

In figure 3 we present the seepage from channels having various profiles. In figure 3.a) we considered $\sigma(s) = -\frac{2\pi}{7} + \frac{\pi}{8}\cos s,\ s \in \left[0, \frac{\pi}{2}\right]$, in figure 2.b) we considered $\sigma(s) = -\frac{\pi}{5} + \frac{\pi}{12}\cos s,\ s \in \left[0, \frac{\pi}{2}\right]$, in figures 3.c) and d) we considered $\sigma(s) = 0,\ s \in \left[0, \frac{\pi}{2}\right]$. We notice a very good agreement between the numerical results obtained in 3.c) and the anlytical results ontained in 3.d) (by means of formulas (6.29) - (6.36) for $\sigma = 0$.

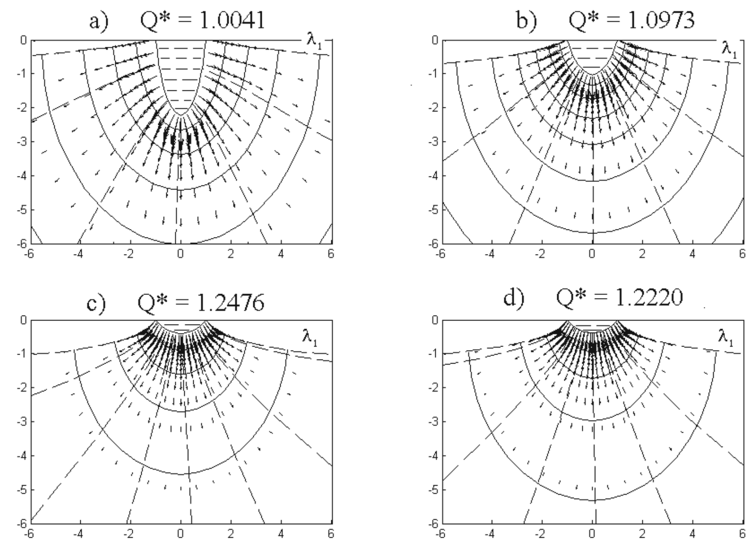

Figure 3. Seepage from various channels. Impermeable layer at infinity

Acknowledgments

The paper was supported by the ECOSCID project of the University of Bucharest. The author is grateful to Professor A. R. Kacimov for his comments and suggestions concerning the paper.

References

[1] K.N. ANAKHAEV, Free percolation and seepage flows from watercourses, *J. Fluid Dyn.*, **39**, *5* (2004), 756-761.

[2] K.N. ANAKHAEV, Calculation of free seepage from watercourses with curvilinear profiles, *Water Resour.*, **34**, *3* (2007), 295-300.

[3] A. CARABINEANU, Numerical method of drainage of underground water by narrow rectangular channels, *Bul. Ştiinţ. Univ. Piteşti Ser. Mat. Inf.*, **9** (2003), 53-64.

[4] A. CARABINEANU and A. HONŢUŞ, *The study of the drainage of underground water by means of the complex boundary intergrals method*, Proc. 2-nd Workshop on Math. Modeling of Environmental and Life Sciences, Bucureşti, sept. 2003, Series on Mathematical Modelling of Environmental and Life Sciences Problems, Ed. Acad. Române, Eds. I. Stelian, G. Marinoschi, 2004, pp. 23-37.

[5] A. CARABINEANU, A free boundary problem in porous media, *Rev. Roumaine Math. Pures Appl.*, **37** (1992), 563-570.

[6] B.R. CHAHAR, Analytical solution to seepage problem from a soil channel with a curvilinear bottom, *Water Resour. Res.*, **42** (2006), 1-10.

[7] B.R. CHAHAR, Optimal design of a special class of curvilinear bottomed channel section, *J. Hydraul. Eng.*, **133**, *5* (2007), 571-576.

[8] M. CHIPOT, *Variational inequalities and flow in porous media*, Applied Mathematical Sciences, vol. 52, Springer-Verlag, New York, 1984.

[9] M. CHIPOT and A. LYAGFOURI, *On the uniqueness of the solution of the dam problem with leaky boundary conditions*, Progress in partial differential equations: the Metz surveys, 4, Pitman Res. Notes Math. Ser., vol. 345, pp. 175-186, Longman, Harlow, 1996.

[10] M. CHIPOT and A. LYAGFOURI, The dam problem for non-linear Darcy's laws and non-linear leaky boundary conditions, *Math. Methods Appl. Sci.*, **20**, *12* (1997), 1045-1086.

[11] A. EL NIMR, Seepage from parallel trapezoidal channels, *J. Eng. Mech. Div. ASCE*, **89**, *4* (1963), 1-10.

[12] P.W. FRANCE, C.J. PAREKH, J.C. PETERS and C. TAYLOR, Numerical analysis of free surface seepage problems, *J. Irr. Drain. Div. ASCE*, **97** (1971), 165-179.

[13] M.E. HARR, *Groundwater and seepage*, McGraw-Hill, New York, 1962.

[14] N.B. ILYINSKY and A.R. KACIMOV, An inverse problem of filtration from a channel in the presence of backwater (Russian), *Trudy Sem. Kraev. Zadacham*, **20** (1984), 104-115.

[15] N.B. ILYINSKY, A.R. KACIMOV and N.D. YAKIMOV, Analytical solutions of seepage theory problems. Inverse method, variational theorems, optimization and estimates (a review), *Fluid Dyn.*, **33**, *2* (1998), 157-168.

[16] A.R. KACIMOV, Discussion of 'Design of minimum seepage loss nonpolygonal canal sections' by Prabhata K. Swamee and Deepak Kashyap, *J. Irrig. Drain. E. ASCE*, **129**, *1* (2003), 68-70.

[17] A.R. KACIMOV and Y.V. OBSONOV, Analytical determination of seeping soil slopes of a constant exit gradient, *Z. Angew. Math. Mech.*, **82**, *6* (2002), 363-376.

[18] J.T. ODEN and N. KIKUCHI, Theory of variational inequalities with applications to problems of flow through porous media, *Int. J. Eng. Sci.*, **18** (1980), 1173-1284.

[19] N.N. PAVLOVSKII, *Dvizhenie gruntovykh vod (Groundwater Flow)*, Akad. Nauk SSSR, Moscow, 1956.

[20] P.Y. POLUBARINOVA-KOCHINA, *Theory of groundwater movement*, Princeton Univ. Press, Princeton, N. J., 1962.

[21] A. PREISSMANN, A propos de la filtration au-dessous des canaux, *La Huile Blanche*, **12**, *2* (1957), 181-190.

[22] V.V. VEDERNIKOV, *Teoriya fil'tratsii i ee primenenie v oblasti irrigatsia i drenazha (Seepage theory and its application in the field of irrigation and drainage)*, Gosstroiizdat, Moscow, 1939.

Adrian Carabineanu

Institute of Mathematical Statistics and Applied Mathematics of Romanian Academy

13 Calea 13 Septembrie, 050711 Bucharest, Romania

and

University of Bucharest, Faculty of Mathematics and Computer Science

14 Academiei Street, 010014 Bucharest, Romania

E-mail: `acara@fmi.unibuc.ro`