Annals of the University of Bucharest (mathematical series)
(Analele Universităţii Bucureşti. Matematică)
1 (LIX) (2010), 165–177

Inequalities for strict bicontractions

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Communicated by George Dinca

Dedicated to Professor Ion Colojoară on his 80th anniversary

Abstract - Versions of some results of C. Foiaş and K. Fan concerning Harnack inequalities for strict contractions are obtained in the context of pairs of commuting contractions on a Hilbert space. The quoted theorems are related to von Neumann's inequality and to Schwarz's lemma for such pairs of contractions. Certain Harnack inequalities for analytic functions in bidisk and for pairs of commuting strict contractions are derived, by using an operatorial version of the maximum modulus principle for analytic functions in bidisk.

Key words and phrases : analytic function, pair of commuting contractions, von Neumann inequality, Harnack inequality, Schwarz's lemma.

Mathematics Subject Classification (2000) **:** 47A20, 47A45, 47A63.

1. Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ with the unit element $I = I_{\mathcal{H}}$.

A contraction on $\mathcal{H}$ is an operator $T \in \mathcal{B}(\mathcal{H})$ with $||T|| \leq 1$, and T is a strict contraction when $||T|| < 1$. Also, an operator $T \in \mathcal{B}(\mathcal{H})$ is strictly positive, and will be denoted $T > 0$, if T is positive and invertible in $\mathcal{B}(\mathcal{H})$. Clearly, $I - T^*T > 0$ if and only if $||T|| < 1$.

The theory of contractions was developed in [16], having as a starting point the classical inequality of von Neumann. Namely, this inequality says that

$$||f(T)|| \leq \sup_{|\lambda|=1} |f(\lambda)| \tag{1.1}$$

for every contraction T on $\mathcal{H}$, and any function f belonging to the disc algebra $A(\mathbb{D})$ of all (complex) continuous functions on $\overline{\mathbb{D}}$ which are analytic on the open unit disc $\mathbb{D}$.

K. Fan [5] proved that the inequality (1.1) is equivalent to the fact that $||f(T)|| < 1$ when $||T|| < 1$ and f is an analytic function on $\mathbb{D}$ with $f(\mathbb{D}) \subset \mathbb{D}$.

This last result may be also derived from Harnack inequalities for strict contractions, as it was showed by C. Foiaş (see [8]).

In this note we give some results for bicontractions, that is for commuting pairs of contractions, which are similar to certain facts of Foiaş [8] and Fan [5, 6, 7] for contractions.

Denote by $A(\mathbb{D}^2)$ the bidisc algebra of all continuous functions on $\overline{\mathbb{D}}^2$ which are analytic on $\mathbb{D}^2$, and by $H^\infty(\mathbb{D}^2)$ the bounded analytic functions on $\mathbb{D}^2$. Denote by $\mathbb{T}$ the unit circle in the complex plane.

Recall that Ando's dilation theorem (see [2], see also [17]) ensures the following version of von Neumann inequality for bicontractions.

Theorem 1.1. *For every bicontraction (T_0, T_1) on $\mathcal{H}$ and $f \in A(\mathbb{D}^2)$ one has*

$$||f(T_0, T_1)|| \leq \sup_{w \in \mathbb{T}^2} |f(w)|. \tag{1.2}$$

In the sequel we obtain some equivalent versions of Theorem 1.1 for bicontractions, and thus we extend some corresponding results from [5, 6, 7] and [8] for contractions. Among others, we prove a version of Schwarz's lemma and certain Harnack type inequalities for strict bicontractions. Such inequalities refer to operator analytic functions, as well as to the functional calculus $f(T)$ for a strict bicontraction $T = (T_0, T_1)$ which means $||T_j|| < 1$, $j = 0, 1$, where the function f is analytic on $\mathbb{D}^2$. Clearly, in this case $f(T)$ can be defined by the Taylor series of f on $\mathbb{D}^2$. We refer to Rudin's book [13] for the theory of analytic functions on the bidisk $\mathbb{D}^2$.

2. Versions of the von Neumann inequality

We begin with the following main fact.

Theorem 2.1. *If f is an analytic function on $\mathbb{D}^2$ with $f(\mathbb{D}^2) \subset \mathbb{D}$, then for every strict bicontraction (T_0, T_1) on $\mathcal{H}$ we have $||f(T_0, T_1)|| < 1$.*

Proof. Let f and (T_0, T_1) be as above. Choose $\lambda_n \in \mathbb{D}$ with $|\lambda_n| \to 1$ $(n \to \infty)$ and define the functions

$$f_n = \frac{1 + \lambda_n f}{1 - \lambda_n f} \quad (n \geq 1).$$

Then f_n is analytic and $\mathrm{Re} f_n \geq 0$, on $\mathbb{D}^2$. On the other hand, for $0 < r < 1$ we have $f_r = f(r\,\cdot) \in A(\mathbb{D}^2)$ and $||f_r|| \leq 1$, therefore $||f_r(T_0, T_1)|| \leq 1$ by the inequality (1.2). Letting $r \to 1$ we get $||f(T_0, T_1)|| \leq 1$, and since $|\lambda_n| < 1$ we infer that $I - \lambda_n f(T_0, T_1)$ is invertible in $\mathcal{B}(\mathcal{H})$. As $f_n(1 - \lambda_n f) = 1 + \lambda_n f$ we obtain

$$f_n(T_0, T_1) = (I + \lambda_n f(T_0, T_1))(I - \lambda_n f(T_0, T_1))^{-1}.$$

Suppose that $||f(T_0,T_1)|| = 1$, and let us consider $h_n \in \mathcal{H}$ with $||h_n|| = 1$ for any $n \geq 1$ such that $||f(T_0,T_1)h_n|| \to 1$, $n \to \infty$. We put

$$k_n = (I - \lambda_n f(T_0,T_1))h_n, \quad k_n^0 = (1 - \lambda_n f(0,0))^{-1} k_n \quad (n \geq 1),$$

and so we get

$$\begin{aligned} \langle \mathrm{Re} f_n(T_0,T_1)k_n, k_n\rangle &= \mathrm{Re}\langle (I + \lambda_n f(T_0,T_1))h_n, (I - \lambda_n f(T_0,T_1))h_n\rangle \\ &= 1 - |\lambda_n|^2 ||f(T_0,T_1)h_n||^2 \to 0, \quad n \to \infty. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \langle \mathrm{Re} f_n(0,0)k_n, k_n\rangle &= \mathrm{Re}\langle (1 + \lambda_n f(0,0))k_n^0, (1 - \lambda_n f(0,0))k_n^0\rangle \\ &= (1 - |\lambda_n f(0,0)|^2)||k_n^0||^2. \end{aligned}$$

Since (T_0,T_1) is a strict bicontraction, it is Harnack equivalent to the null bicontraction $(0,0)$ (see [15, Theorem 3.6]), hence there exists a constant $c \geq 1$ such that

$$\mathrm{Re} f_n(0,0)I \leq c\mathrm{Re} f_n(T_0,T_1)$$

for any $n \geq 1$. This inequality and the previous relations imply

$$(1 - |\lambda_n f(0,0)|^2)||k_n^0||^2 \to 0, \quad n \to \infty.$$

Since $|\lambda_n| < 1$ for any n and $|f(0,0)| < 1$ we have

$$0 < (1 - |f(0,0)|^2)||k_n^0||^2 \leq (1 - |\lambda_n f(0,0)|^2)||k_n^0||^2.$$

Hence $k_n \to 0$, for $n \to \infty$. Finally, having in view the expression of k_n, and taking $\lambda_n = 1 - \frac{1}{n}$ and later $\lambda_n = -1 + \frac{1}{n}$, we get $h_n \to 0$ $(n \to \infty)$, which contradicts the fact that $||h_n|| = 1$ for any n. So $||f(T_0,T_1)|| < 1$ and the proof is finished. □

Remark 2.1. From the previous proof we infer that Theorem 1.1 implies Theorem 2.1. Conversely, let $f \in A(\mathbb{D}^2)$, $f \neq 0$ and (T_0,T_1) be a bicontraction on $\mathcal{H}$. For every $r \in (0,1)$ we have by Theorem 2.1 that $||g(rT_0,rT_1)|| < 1$ where $g = \frac{f}{||f||}$, since $g(\mathbb{D}^2) \subset \mathbb{D}$ by the maximum modulus principle. Letting $r \to 1$ we get $||g(T_0,T_1)|| \leq 1$, or equivalently $||f(T_0,T_1)|| \leq ||f||$. So Theorem 1.1 follows from Theorem 2.1.

Remark also that Theorem 2.1 is the version for bicontractions of the corresponding result of K. Fan [5] for contractions, but our proof is different from the one of [5]. Here we used an argument inspired by C. Foiaş (see [8]).

Another version for bicontractions of a corresponding result of [5] for contractions, is the following form of the maximum modulus principle for analytic functions on $\mathbb{D}^2$.

Theorem 2.2. *For every analytic function f on $\mathbb{D}^2$ and $0 \leq r < 1$ we have*

$$\sup_{||T_j|| \leq r} ||f(T_0, T_1)|| = \sup_{|z_0|=|z_1|=r} |f(z_0, z_1)|, \tag{2.1}$$

where the supremum in the left side of (2.1) is taken over all bicontractions (T_0, T_1) on $\mathcal{H}$ with $||T_j|| \leq r$, $j = 0, 1$.

Proof. Let $f \neq 0$ be as above and $0 < r < 1$. We denote

$$||f||_r = \sup_{|z_0|=|z_1|=r} |f(z_0, z_1)|,$$

so $||f||_r \neq 0$. Then the function $g = \frac{f_r}{||f||_r}$ belongs to $A(\mathbb{D}^2)$ and $||g|| = 1$, while by (1.2) we have $||g(T_0, T_1)|| \leq 1$ for every bicontractions (T_0, T_1) on $\mathcal{H}$. So, if $||T_j|| \leq r$, $j = 0, 1$, we obtain

$$||f(T_0, T_1)|| = ||f_r(\frac{1}{r}T_0, \frac{1}{r}T_1)|| \leq ||f||_r,$$

hence

$$\sup_{||T_j|| \leq r} ||f(T_0, T_1)|| \leq ||f||_r,$$

which gives an inequality in (2.1). The converse inequality is trivial: if $w_j \in \mathbb{D}$ with $|w_j| = r$ and $|f(w_0, w_1)| = ||f||_r$ then

$$||f||_r = ||f(w_0 I, w_1 I)|| \leq \sup_{||T_j|| \leq r} ||f(T_0, T_1)||.$$

This ends the proof. □

Remark 2.2. As we saw, the previous result is a direct consequence of Theorem 1.1. Furthermore, Theorem 2.2 clearly implies Theorem 2.1. Indeed, if f is an analytic function on $\mathbb{D}^2$ with $f(\mathbb{D}^2) \subset \mathbb{D}$ and (T_0, T_1) is a strict bicontraction on $\mathcal{H}$, then applying Theorem 2.2 with $r = \max(||T_0||, ||T_1||) < 1$, we get

$$||f(T_0, T_1)|| \leq ||f||_r < 1,$$

that is the conclusion of Theorem 2.1.

Another result closely related to the above theorems is the following:

Theorem 2.3. *Let f, g, l be analytic functions on $\mathbb{D}^2$ such that $f = lg$ and $|l(z)| \leq 1$ for $z \in \mathbb{D}^2$. Then for every strict bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ we have*

$$f(T)^* f(T) \leq g(T)^* g(T) \tag{2.2}$$

and

$$||f(T)|| \leq ||g(T)||. \tag{2.3}$$

Moreover, the strict inequality holds in (2.2) if and only if $g(T)^*g(T)$ *is strictly positive and* l *is not a constant function of modulus* 1*. Also, the equality in (2.3) occurs if and only if either* $g(T) = 0$*, or* l *is a constant function with* $|l| = 1$*.*

Proof. We can assume that l is not reduced to a constant λ with $|\lambda| = 1$ (otherwise, we have the equality in both relations (2.2) and (2.3)). Then by hypothesis and the maximum modulus principle one has $|l(z)| < 1$ for $z \in \mathbb{D}^2$, while by Theorem 2.1 we have $||l(T)|| < 1$ for every bicontraction $T = (T_0, T_1)$ with $||T_j|| < 1$, $j = 0, 1$. Since $f = lg$ we get $f(T) = l(T)g(T)$ and

$$f(T)^*f(T) = g(T)^*l(T)^*l(T)g(T) \leq ||l(T)||^2 g(T)^*g(T).$$

If follows that

$$g(T)^*g(T) - f(T)^*f(T) \geq (1 - ||l(T)||^2)g(T)^*g(T), \tag{2.4}$$

whence we infer the inequalities (2.2) and (2.3). When $g(T)^*g(T) > 0$ (a positive invertible operator), the inequality (2.4) leads to the strict inequality in (2.2), and conversely, when the inequality (2.2) is strict one has

$$g(T)^*g(T) \geq g(T)^*g(T) - f(T)^*f(T) > 0.$$

Now, if $||f(T)|| = ||g(T)||$ then $||f(T)|| \leq ||l(T)||\,||f(T)||$, hence

$$||f(T)||(1 - ||l(T)||) \leq 0.$$

This yields either $f(T) = 0$, or $1 - ||l(T)|| \leq 0$, while the last inequality implies (by Theorem 2.1 and the maximum modulus principle) that $l(z) = \lambda$ for $z \in \mathbb{D}^2$ with $|\lambda| = 1$. Conversely, if either l has such a form or $g(T) = 0$, then clearly $||f(T)|| = ||g(T)||$. This ends the proof. □

Remark that we used Theorem 2.1 to obtain Theorem 2.3, but Theorem 2.1 may be also regarded as a special case ($g = 1$) of the last theorem. Thus, Theorems 2.1, 2.2 and 2.3 are versions of Theorem 1.1 for bicontractions. These facts involving strict bicontractions, are similar to the corresponding results of K. Fan (see [5]) for strict contractions.

Notice from [14] and [15] that Theorem 2.1 can be also obtained from Harnack inequalities for strict bicontractions, and that from Theorem 2.3 we can derived some Harnack type inequalities for such bicontractions, similar to those from [5] for strict contractions, or more general, for strict ρ-contractions which were obtained in [3, 4].

As another application of Theorem 2.3, we deduce now the following operator analogue of Schwarz's lemma on bidisk. Recall that an extension of this lemma for contractions was given in [5], while the version for ρ-contractions appears in [3, 4].

Proposition 2.1. *Let $f \in H^\infty(\mathbb{D}^2)$ with $||f_\infty|| \leq 1$. Then for any strict bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$ we have*

$$||\frac{\partial f}{\partial z_0}(T) + \frac{\partial f}{\partial z_1}(T)|| \leq \frac{||I - f(T)^2||}{1 - \max_{j=0,1} ||T_j||^2}. \tag{2.5}$$

Proof. It is known (see [1], [13]) that Schwarz's lemma on $\mathbb{D}^2$ says that for $z = (z_0, z_1) \in \mathbb{D}^2$ one has

$$(1 - |z_0|^2)|\frac{\partial f}{\partial z_0}(z)| + (1 - |z_1|)|\frac{\partial f}{\partial z_1}(z)| \leq 1 - |f(z)|^2. \tag{2.6}$$

Let $T = (T_0, T_1)$ be a strict bicontraction on $\mathcal{H}$ and $r = \max_{j=0,1} ||T_j||$. Then for $z = (z_0, z_1)$ with $||z_j|| = r$, $j = 0, 1$ one obtains

$$(1 - r^2)(|\frac{\partial f}{\partial z_0}(z)| + |\frac{\partial f}{\partial z_1}(z)|) \leq |1 - f(z)^2|,$$

while by Theorem 2.2 we get

$$(1 - r^2)||(\frac{\partial f}{\partial z_0}(T) + \frac{\partial f}{\partial z_1}(T))(I - f(T)^2)^{-1}|| \leq$$

$$\sup_{|z_j|=r} (1 - r^2)\frac{|\frac{\partial f}{\partial z_0}(z) + \frac{\partial f}{\partial z_1}(z)|}{|1 - f(z)^2|} \leq 1.$$

Note that by the maximum modulus principle we have $f(\mathbb{D}^2) \subset \mathbb{D}$ and so $||f(T)|| < 1$ by Theorem 2.1, if f is not reduced to a constant. The above inequality leads to

$$(1 - r^2)||\frac{\partial f}{\partial z_0}(T) + \frac{\partial f}{\partial z_1}(T)|| \leq ||I - f(T)^2||,$$

which means (2.5). This ends the proof. □

Notice that this result is a weaker form of Schwarz's lemma because for $f \in H^\infty(\mathbb{D}^2)$ with $||f||_\infty \leq 1$ we only infer from (2.5) that $|\frac{\partial f}{\partial z_0}(0) + \frac{\partial f}{\partial z_1}(0)| \leq |1 - f(0)^2|$ (to compare with (2.6) for $z = 0$).

Remark 2.3. From the inequality (2.6) it follows by Theorem 2.2 that (for f and T as in Proposition 2.1)

$$||f(T)^2 + (1 - \max_{j=0,1} ||T_j||^2)(\frac{\partial f}{\partial z_0}(T) + \frac{\partial f}{\partial z_1}(T))|| \leq 1, \tag{2.7}$$

but we do not know if the above contraction is a strict contraction.

3. Harnack type inequalities

It is well known from [5] that the version of Theorem 2.3 for the functions of a single variable leads to operator analogue of Schwarz's lemma, and of the classical Harnack and Carathéodory inequalities. These facts are essentially based on Schwarz's lemma, concerning the factorization of a contractive analytic function f on $\mathbb{D}$ with $f(0) = 0$ by the function $u(z) = z^n$, where n is the order of multiplicity of $z = 0$ for f. A similar factorization fails for the functions in $H^\infty(\mathbb{D}^2)$ (see [1], [13]), and even the role of Blaschke products cannot be "well-defined" in this context.

However, for the functions in $H^\infty(\mathbb{D}^2)$ which have the same zeros with certain inner functions, we can obtain some Harnack type inequalities for strict bicontractions.

Recall that $f \in H^\infty(\mathbb{D}^2)$ is an inner function if $|f^*| = 1$ a.e. on $\mathbb{T}^2$ relative to the (normalized) Lebesgue measure m_2 on $\mathbb{T}^2$, where f^* is the radial function of f.

An inner function $f \in H^\infty(\mathbb{D}^2)$ is called a *good inner function* if the least 2-harmonic majorant for the function $\log|f|$ is the null function. Equivalently, this means that for a.e. (m_2) $w \in \mathbb{T}^2$, the function $\hat{f_w}$ is a Blaschke product on $\mathbb{D}$, where $\hat{f_w}(\lambda) = f(\lambda w)$, $\lambda \in \mathbb{D}$ (see [13]).

The fact that two functions $f, f_0 \in H^\infty(\mathbb{D}^2)$ have the same zeros, means that there exists an analytic function g on $\mathbb{D}^2$ which has no zeros in $\mathbb{D}^2$, such that $f = f_0 g$. In this case, we also say that f can be factorized by f_0.

Now, we can obtain some Harnack type inequalities for strict bicontractions, similar to those of [5, 6, 7] for strict contractions.

Theorem 3.1. *Let $f \in H^\infty(\mathbb{D}^2)$ with $\mathrm{Re} f(z) > 0$ for $z \in \mathbb{D}^2$, such that $f - 1$ has the same zeros like a good inner function f_0 in $\mathbb{D}^2$. Then for any strict bicontraction $T = (T_0, T_1)$ on $\mathcal{H}$, the following inequalities hold:*

$$[I - f(T)^*][I - f(T)] \le [I + f(T)^*] f_0(T)^* f_0(T) [I + f(T)], \tag{3.1}$$

$$\frac{1 - ||f_0(T)||}{1 + ||f_0(T)||} \le ||f(T)|| \le \frac{1 + ||f_0(T)||}{1 - ||f_0(T)||}, \tag{3.2}$$

$$\frac{1 - ||f_0(T)||}{1 + ||f_0(T)||} I \le \mathrm{Re} f(T) \le \frac{1 + ||f_0(T)||}{1 - ||f_0(T)||} I, \tag{3.3}$$

$$\frac{-2||f_0(T)||}{1 - ||f_0(T)||^2} I \le \mathrm{Im} f(T) \le \frac{2||f_0(T)||}{1 - ||f_0(T)||^2} I. \tag{3.4}$$

Strict inequality in (3.1) occurs if and only if $f_0(T)^* f_0(T) > 0$ and f fails to have the form $f = (1 + \lambda f_0)(1 - \lambda f_0)^{-1}$ for some scalar λ with $|\lambda| = 1$.

Proof. We consider the function $g = (f-1)(f+1)^{-1} \in H^\infty(\mathbb{D}^2)$, which has the same zeros like $f-1$ and so, as f_0 in $\mathbb{D}^2$. Hence there exists an analytic function f_1 on $\mathbb{D}^2$ such that $f_1(z) \neq 0$, $z \in \mathbb{D}^2$ and $g = f_0 f_1$. Since f_0 is a good inner function, by Theorem 5.4.2 [13] we have $f_1 \in H^\infty(\mathbb{D}^2)$ and $||f_1||_\infty = ||g||_\infty \leq 1$. Thus, if $T = (T_0, T_1)$ is a strict bicontraction on $\mathcal{H}$, by Theorem 2.3 one has

$$g(T)^* g(T) \leq f_0(T)^* f_0(T),$$

or equivalently,

$$[I + f(T)^*]^{-1}[I - f(T)^*][I - f(T)][I + f(T)]^{-1} \leq f_0(T)^* f_0(T).$$

Clearly, this relation can also be written in the form (3.1). In addition, we infer from Theorem 2.3 that the strict inequality in (3.1) occurs if and only if $f_0(T)^* f_0(T) > 0$ and the above function f_1 is not reduced to a constant λ with $|\lambda| = 1$, or in other words, f is not of the form $f = (1+\lambda f_0)(1-\lambda f_0)^{-1}$ with $|\lambda| = 1$.

Next, we obtain from (3.1)

$$||f(T)|| - 1 \leq ||I - f(T)|| \leq ||f_0(T)||(1 + ||f(T)||),$$

or equivalently,

$$||f(T)||(1 - ||f_0(T)||) \leq 1 + ||f_0(T)||.$$

Since f_0 is an inner function, by the maximum modulus principle one has $|f_0(z)| < 1$ for $z \in \mathbb{D}^2$, and by Theorem 2.1 we have $||f_0(T)|| < 1$. So, the previous inequality gives the inequality on the right of (3.2).

Now, the assumption on f implies $f(z) \neq 0$, $z \in \mathbb{D}^2$, and also $f^{-1} - 1$ has the same zeros like f_0. Then by the above remark we get

$$\frac{1}{||f(T)||} \leq ||f^{-1}(T)|| \leq \frac{1 + ||f_0(T)||}{1 - ||f_0(T)||},$$

whence it follows the inequality on the left of (3.2).

It is easy to see that both inequalities (3.2) are equivalent to the following inequality

$$||f(T) - \frac{1 + ||f_0(T)||^2}{1 - ||f_0(T)||^2} I|| \leq \frac{2||f_0(T)||}{1 - ||f_0(T)||^2}. \tag{3.5}$$

Since we have

$$\pm \mathrm{Re}[f(T) - \frac{1 + ||f_0(T)||^2}{1 - ||f_0(T)||^2} I] \leq ||f(T) - \frac{1 + ||f_0(T)||^2}{1 - ||f_0(T)||^2} I||,$$

we infer the inequalities (3.3) from (3.5). Also (3.5) implies (3.4) because

$$\pm \mathrm{Im} f(T) = \pm \mathrm{Im}[f(T) - \frac{1 + ||f_0(T)||^2}{1 - ||f_0(T)||^2} I] \leq ||f(T) - \frac{1 + ||f_0(T)||^2}{1 - ||f_0(T)||^2} I||.$$

The proof is finished. □

Remark 3.1. Suppose $f \in H^\infty(\mathbb{D}^2)$ with $\mathrm{Re} f(z) > 0$ for $z \in \mathbb{D}^2$ such that $f - \mathrm{Re} f(0)$ has the same zeros as a good inner function f_0 in $\mathbb{D}^2$. Then all inequalities (3.1)-(3.5) hold also for the function $f[\mathrm{Re} f(0)]^{-1}$ instead of f. This shows that, for instance, the corresponding inequalities (3.3) obtained in this case are some generalizations for bidisk, and in the context of bicontractions, of the classical Harnack's inequalities. But the inequalities given by Theorem 3.1 only refer to the functions in $H^\infty(\mathbb{D}^2)$ which can be factorized by good inner functions.

Remark 3.2. A good inner function f_0 having the same zeros like a function $f \in H^\infty(\mathbb{D}^2)$, if it exists, is uniquely determined up to a unimodular constant. Recall ([13], Theorem 5.4.3) that there exists such a function f_0 for f, if the least 2-harmonic majorant for $\log |f|$ is the real part of an analytic function on $\mathbb{D}^2$; in particular, this happens when the quoted majorant is just Poisson integral of the function $\log |f^*|$ ([13], Theorem 5.4.6 and Theorem 5.4.7). But in general, there is $f \in H^\infty(\mathbb{D}^2)$ such that f cannot be factorized by inner functions ([13], Theorem 5.4.8). By contrast, any function $0 \neq f \in A(\mathbb{D}^2)$ has the same zeros like an inner function, but not necessary a good inner function ([13], Theorem 5.4.5).

If $f \in H^\infty(\mathbb{D})$ then f has a good inner factor f_0 which is a Blaschke product ([13], Theorem 5.3.2). By Schwarz's lemma f can be factorized by any Blaschke factor which gives a zero of f. So, if $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = f'(0) = ... = f^{(n-1)}(0) = 0$ for some integer $n \geq 1$, then $f_0(z) = z^n f_1(z)$, $z \in \mathbb{D}$, where $f_1 \in H^\infty(\mathbb{D})$ and $|f_1| \leq 1$. Then for any strict contraction T on $\mathcal{H}$ one has $||f_0(T)|| \leq ||T^n||$, and thus we can deduce similar inequalities to (3.1)-(3.5) with T^n (T a strict contraction) instead of $f_0(T)$ from above. Such inequalities are given in [5, Corollary 3] and [6, Proposition 2].

Recently (see [15]) we obtained some Harnack inequalities for the functional calculus with strict bicontractions and with analytic functions on bidisk, without the hypothesis of factorization by good inner functions. We see now these inequalities for operator-valued analytic functions on $\mathbb{D}^2$.

Theorem 3.2. *Let $\Theta : \mathbb{D}^2 \to \mathcal{B}(\mathcal{H})$ be an analytic function on $\mathbb{D}^2$ such that $\mathrm{Re}\Theta(z) > 0$ for any $z \in \mathbb{D}^2$. Then the inequalities*

$$\frac{(1-|z_0|)(1-|z_1|)}{(1+|z_0|)(1-|z_1|)}\mathrm{Re}\Theta(0) \leq \mathrm{Re}\Theta(z) \leq \frac{(1+|z_0|)(1+|z_1|)}{(1-|z_0|)(1-|z_1|)}\mathrm{Re}\Theta(0), \quad (3.6)$$

and

$$\mathrm{Im}\Theta(0) - \frac{2(|z_0| + |z_1| + 2|z_0 z_1|)}{(1-|z_0|^2)(1-|z_1|^2)}\mathrm{Re}\Theta(0) \leq \mathrm{Im}\Theta(z) \leq \quad (3.7)$$

$$\leq \frac{2(|z_0| + |z_1| + 2|z_0 z_1|)}{(1-|z_0|^2)(1-|z_1|^2)}\mathrm{Re}\Theta(0) + \mathrm{Im}\Theta(0)$$

hold for all $z = (z_0, z_1) \in \mathbb{D}^2$.

Proof. We use Proposition 3 from [7], which claims that if F is a $\mathcal{B}(\mathcal{H})$-valued analytic function on $\mathbb{D}$ with $\mathrm{Re}F(\lambda) > 0$ for $\lambda \in \mathbb{D}$ and $F(0) = I$, then the following inequalities hold:

$$\frac{1-|\lambda|}{1+|\lambda|}I \leq \mathrm{Re}F(\lambda) \leq \frac{1+|\lambda|}{1-|\lambda|}I, \quad \frac{-2|\lambda|}{1-|\lambda|^2}I \leq \mathrm{Im}F(\lambda) \leq \frac{2|\lambda|}{1-|\lambda|^2}I. \tag{3.8}$$

In the case $F(0) \neq I$ we have $F(0)h \neq 0$ for any $h \in \mathcal{H}$, $h \neq 0$, because $F(0)$ is a positive invertible operator. In this case, we can consider the function F_h on $\mathbb{D}$ given by

$$F_h(\lambda) = \frac{\langle (F(\lambda) - i\mathrm{Im}F(0))h, h\rangle}{\langle \mathrm{Re}F(0)h, h\rangle} \quad (\lambda \in \mathbb{D})$$

which is analytic in $\mathbb{D}$ with $\mathrm{Re}F_h(\lambda) > 0$ and $F_h(0) = 1$. Applying (3.8) and having in view that $h \in \mathcal{H}$ is arbitrary, we obtain

$$\frac{1-|\lambda|}{1+|\lambda|}\mathrm{Re}F(0) \leq \mathrm{Re}F(\lambda) \leq \frac{1+|\lambda|}{1-|\lambda|}\mathrm{Re}F(0)$$

and

$$\mathrm{Im}F(0) - \frac{2|\lambda|}{1-|\lambda|^2}\mathrm{Re}F(0) \leq \mathrm{Im}F(\lambda) \leq \frac{2|\lambda|}{1-|\lambda|^2}\mathrm{Re}F(0) + \mathrm{Im}F(0).$$

Let Θ be as in hypothesis. For $w \in \mathbb{D}$ we define the functions $\Theta_w^0, \Theta_w^1 : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ by $\Theta_w^0(\lambda) = \Theta(w, \lambda)$, $\Theta_w^1(\lambda) = \Theta(\lambda, w)$, $\lambda \in \mathbb{D}$. Then Θ_w^0, Θ_w^1 are analytic on $\mathbb{D}$ with $\mathrm{Re}\Theta_w^j(\lambda) > 0$ for $\lambda \in \mathbb{D}$ and $j = 0, 1$. Thus, by the previous inequalities for Θ_w^0 and Θ_w^1 we have, successively, for any $z = (z_0, z_1) \in \mathbb{D}^2$

$$\begin{aligned} \mathrm{Re}\Theta(z) = \mathrm{Re}\Theta_{z_1}^1(z_0) &\leq \frac{1+|z_0|}{1-|z_0|}\mathrm{Re}\Theta(0, z_1) = \frac{1+|z_0|}{1-|z_0|}\mathrm{Re}\Theta_0^0(z_1) \\ &\leq \frac{1+|z_0|}{1-|z_0|} \cdot \frac{1+|z_1|}{1-|z_1|}\mathrm{Re}\Theta(0), \end{aligned}$$

and respectively

$$\mathrm{Re}\Theta(z) \geq \frac{1-|z_0|}{1+|z_0|}\mathrm{Re}\Theta_0^0(z_1) \geq \frac{1-|z_0|}{1+|z_0|} \cdot \frac{1-|z_1|}{1+|z_1|}\mathrm{Re}\Theta(0).$$

In a similar way, we find

$$\begin{aligned} \mathrm{Im}\Theta(z) &= \mathrm{Im}\Theta_{z_0}^0(z_1) \leq \frac{2|z_1|}{1-|z_1|^2}\mathrm{Re}\Theta(z_0, 0) + \mathrm{Im}\Theta(z_0, 0) \leq \\ &\leq \frac{2|z_1|}{1-|z_1|^2} \cdot \frac{1+|z_0|}{1-|z_0|}\mathrm{Re}\Theta(0,0) + \frac{2|z_0|}{1-|z_0|^2}\mathrm{Re}\Theta(0) + \mathrm{Im}\Theta(0) \\ &= \frac{2[|z_0| + |z_1| + |z_0 z_1|(2 + |z_0| - |z_1|)]}{(1-|z_0|^2)(1-|z_1|^2)}\mathrm{Re}\Theta(0) + \mathrm{Im}\Theta(0). \end{aligned}$$

Note that if we consider firstly $\Theta(z) = \Theta^1_{z_1}(z_0)$, we obtain a similar inequality where the term $|z_1| - |z_0|$ appears to the coefficient of $\mathrm{Re}\Theta(0)$. Since $|z_j| - |z_{1-j}| \leq 0$ for $j = 0$ or $j = 1$ and $\mathrm{Re}\Theta(0) > 0$, we infer that

$$\mathrm{Im}\Theta(z) \leq \frac{2(|z_0| + |z_1| + 2|z_0 z_1|)}{(1-|z_0|^2)(1-|z_1|^2)}\mathrm{Re}\Theta(0) + \mathrm{Im}\Theta(0).$$

By symmetry, we also have

$$\begin{aligned}
\mathrm{Im}\Theta(z) &\geq \mathrm{Im}\Theta^0_{z_0}(0) - \frac{2|z_1|}{1-|z_1|^2}\mathrm{Re}\Theta^0_{z_0}(0) \\
&= \mathrm{Im}\Theta^1_0(z_0) - \frac{2|z_1|}{1-|z_1|^2}\mathrm{Re}\Theta^1_0(z_0) \\
&\geq \mathrm{Im}\Theta(0) - \frac{2|z_0|}{1-|z_0|^2}\mathrm{Re}\Theta(0) - \frac{2|z_1|}{1-|z_1|^2} \cdot \frac{1+|z_0|}{1-|z_0|}\mathrm{Re}\theta(0) \\
&= \mathrm{Im}\Theta(0) - \frac{2[|z_0| + |z_1| + |z_0 z_1|(2 + |z_0| - |z_1|)]}{(1-|z_0|^2)(1-|z_1|^2)}\mathrm{Re}\Theta(0).
\end{aligned}$$

Having in mind the other inequality which one obtains if we firstly consider $\Theta(z) = \Theta^1_{z_1}(z_0)$, we infer as above that

$$\mathrm{Im}\Theta(z) \geq \mathrm{Im}\Theta(0) - \frac{2(|z_0| + |z_1| + 2|z_0 z_1|)}{(1-|z_0|^2)(1-|z_1|^2)}\mathrm{Re}\Theta(0) + \mathrm{Im}\Theta(0).$$

This ends the proof. □

Corollary 3.1. *Let Θ be a $\mathcal{B}(\mathcal{H})$-valued analytic function on $\mathbb{D}^2$ such that* $\mathrm{Re}\Theta(z) > 0$ *for $z \in \mathbb{D}^2$ and $\Theta(0) = I$. Then for $0 < r < 1$ we have*

$$\sup_{|z_j|=r} ||\Theta(z_0, z_1)|| \leq \frac{(1+r)^3 + 4r}{(1-r)^2(1+r)}. \tag{3.9}$$

Proof. If $|z_j| = r$ $(j = 0, 1)$ then the inequalities (3.6) and (3.7) become (because $\Theta(0) = I$)

$$\mathrm{Re}\theta(z_0, z_1) \leq (\frac{1+r}{1-r})^2 I, \quad \pm\mathrm{Im}\theta(z_0, z_1) \leq \frac{4r(1+r)}{(1-r^2)^2} I.$$

These imply the inequality (3.9) because

$$||\Theta(z)|| \leq ||\mathrm{Re}\theta(z)|| + ||\mathrm{Im}\Theta(z)||,$$

and $\mathrm{Re}\Theta(z) > 0$, while $\mathrm{Im}\theta(z)$ is a selfadjoint operator, for any $z \in \mathbb{D}^2$. □

Remark 3.3. Corollary 3.1 yields a complete analogy to the unit disk case. Indeed, if F is a function as in the proof of Theorem 3.2, then it follows from (3.8) that for $0 < r < 1$ one has

$$\sup_{|\lambda|=r} ||F(\lambda)|| \leq \frac{(1+r)^2 + 2r}{1-r^2}.$$

Let us notice that we proved in [15] similar inequalities to (3.6), for the functional calculus $f(T)$ with analytic functions f on $\mathbb{D}^2$ such that $\mathrm{Re} f > 0$, induced by strict bicontractions $T = (T_0, T_1)$ on $\mathcal{H}$. More exactly, we have

$$\frac{(1-||T_0||)(1-||T_1||)}{(1+||T_0||)(1+||T_1||)} \mathrm{Re} f(0) I \leq \mathrm{Re} f(T) \leq \qquad (3.10)$$
$$\leq \frac{(1+||T_0||)(1+||T_1||)}{(1-||T_0||)(1-||T_1||)} \mathrm{Re} f(0) I.$$

This result was obtained using some $\mathcal{B}(\mathcal{H})$-valued semispectral measure canonically attached to T, and of course, we also applied the result of C. Foiaş (see [8]) concerning Harnack inequality for strict contractions. The above inequalities yield immediately that the hyperbolic distance between (T_0, T_1) and $(0, 0)$ is not greater than the sum of hyperbolic distances between T_0 and 0, and between T_1 and 0.

Remark that recently, by a series of three papers, G. Popescu [10, 11, 12] develops a non-commutative hyperbolic geometry on the unit ball of $\mathcal{B}(\mathcal{H})^n$, having as a starting point the operator Harnack inequalities for contractions, and the corresponding hyperbolic metric (see [15]). Harnack inequalities obtained by G. Popescu show, in particular, that if a bicontraction $T = (T_0, T_1)$ satisfies also $||T_0 T_0^* + T_1 T_1^*|| < 1$, then the inequalities (3.6) and (3.10) can be improved by a constant less than the one which appears in the quoted inequalities.

Acknowledgments

The authors are grateful to the referee for a careful reading of the manuscript and for his useful comments which improved the original version.

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