Annals of the University of Bucharest (mathematical series)
(Analele Universităţii Bucureşti. Matematică)
1 (LIX) (2010), 155–164

The stability of some operator equations in Hilbert spaces

Daniel Stănică

Communicated by Romulus Cristescu

To Professor Ion Colojoară on the occasion of his 80th birthday

Abstract - Let $T : X \to Y$ be a continuous linear operator with closed range, where X and Y are Hilbert spaces. In this paper we present some new results concerning of stability analysis for the equation $T(x) = y$ and the least squares equation $\|T(x) - b\| = \inf_{z \in X} \|T(z) - b\|$ with some type perturbations.

Key words and phrases : stability, perturbation analysis, pseudoinverse, condition number.

Mathematics Subject Classification (2000) **:** 47A50, 47A55, 65F20.

1. Introduction

The operator equations and the least squares problems are widely used in various areas of computational and applied mathematics (see, for example, [1]). Hence its stability (or perturbation analysis) is important in error estimate for computing solutions. In this paper we present results for the stability of some operator equation in Hilbert spaces which generalize well-known results for matrix equations and improve some formulas obtained in [2], [3] and [4]. The main tools of our work are the pseudoinverse of a linear continuous operator and an associated condition number. For the theory of pseudoinverse we can see [1].

Let $T : X \to Y$ be a continuous linear operator with closed range, where X and Y are Hilbert spaces. Denote by $Tx = T(x)$, for all $x \in X$ and by $R(T) := \{y \in Y \mid \text{there exists } x \in X \text{ such that } y = Tx\}$ the range of T. Assume that $R(T)$ is a closed subspace in Y. Let $T^+ : Y \to X$ be the pseudoinverse (Moore-Penrose inverse) of T and let us consider the condition number of T given by $\text{cond}(T) := \|T\| \cdot \|T^+\|$.

2. The stability of $Tx = b$ type equation with $Ty = b + \delta b$ type perturbation

We consider the following operator equations:

$$Tx = b \tag{2.1}$$

and

$$Ty = b + \delta b \tag{2.2}$$

with $b, b + \delta b \in R(T), b \neq 0$.

Theorem 2.1. *a) For each solution x of the equation* (2.1) *there exists a solution y_0 of the equation* (2.2) *such that*

$$\frac{\|y_0 - x\|}{\|x\|} \leq \operatorname{cond}(T) \cdot \frac{\|\delta b\|}{\|b\|}.$$

b) For each solution y of the equation (2.2) *there exists a solution x_0 of the equation* (2.1) *such that*

$$\frac{\|y - x_0\|}{\|x_0\|} \leq \operatorname{cond}(T) \cdot \frac{\|\delta b\|}{\|b\|}.$$

Proof. Let us consider $x \in X$ which verifies the equation $Tx = b$ and $z \in N(T)$. We take $y := x + T^+\delta b + z$. Then $Ty = Tx + TT^+\delta b + Tz$. We have $TT^+\delta b = \delta b$, because $\delta b = b + \delta b - b \in R(T)$. It results that $Ty = Tx + \delta b = b + \delta b$. So y verifies the equation $Ty = b + \delta b$.

Let $y_0 := x + T^+\delta b$ (which is the value corresponding to $z = 0$). On one side, $y_0 - x = T^+\delta b$, which implies

$$\|y_0 - x\| \leq \|T^+\|\|\delta b\|.$$

On the other side,

$$\|b\| = \|Tx\| \leq \|T\|\|x\| \Rightarrow \frac{1}{\|x\|} \leq \frac{\|T\|}{\|b\|}.$$

It results that

$$\frac{\|y_0 - x\|}{\|x\|} \leq \|T^+\|\|\delta b\|\frac{\|T\|}{\|b\|} = \operatorname{cond}(T) \cdot \frac{\|\delta b\|}{\|b\|}.$$

b) Let us consider $y \in X$ which verifies the equation $Ty = b + \delta b$ and $z \in N(T)$. We take $x := y - T^+\delta b - z$. Then $Tx = Ty - TT^+\delta b - Tz = b + \delta b - \delta b = b$. So x verifies the equation $Tx = b$.

Let $x_0 := y - T^+\delta b$ (which is the value corresponding to $z = 0$). On one side, $y - x_0 = T^+\delta b$, so

$$\|y - x_0\| \leq \|T^+\|\|\delta b\|.$$

On the other side,

$$\|b\| = \|Tx_0\| \le \|T\|\|x_0\| \Rightarrow \frac{1}{\|x_0\|} \le \frac{\|T\|}{\|b\|}.$$

It results that

$$\frac{\|y - x_0\|}{\|x_0\|} \le \|T^+\|\|\delta b\|\frac{\|T\|}{\|b\|} = \mathrm{cond}(T) \cdot \frac{\|\delta b\|}{\|b\|}.$$

□

Theorem 2.2. *a) For any solution x of the equation* (2.1) *and any solution y of the equation* (2.2) *we have*

$$\frac{\|y - x\|}{d(x, N(T))} \ge \frac{1}{\mathrm{cond}(T)} \cdot \frac{\|\delta b\|}{\|b\|}.$$

b) For any solution y of the equation (2.2) *there exists a solution x_0 of the equation* (2.1) *such that*

$$\frac{\|y - x_0\|}{\|x_0\|} \ge \frac{1}{\mathrm{cond}(T)} \cdot \frac{\|\delta b\|}{\|b\|}.$$

Proof. From $Tx = b$ and $Ty = b + \delta b$ it results that $T(y - x) = \delta b$ and therefore

$$\|y - x\| \ge \frac{\|\delta b\|}{\|T\|}.$$

a) From $Tx = b$ it results that there exists $z \in N(T)$ such that $x = T^+b + z$. Then

$$\|x - z\| = \|T^+b\| \le \|T^+\|\|b\| \Rightarrow$$

$$\frac{1}{\|x - z\|} \ge \frac{1}{\|T^+\|\|b\|} \Rightarrow \frac{1}{d(x, N(T))} \ge \frac{1}{\|T^+\|\|b\|}.$$

Hence

$$\frac{\|y - x\|}{d(x, N(T))} \ge \frac{\|\delta b\|}{\|T\|} \cdot \frac{1}{\|T^+\|\|b\|} = \frac{1}{\mathrm{cond}(T)} \cdot \frac{\|\delta b\|}{\|b\|}.$$

b) Let $x_0 = T^+b$. Then

$$\|x_0\| = \|T^+b\| \le \|T^+\|\|b\| \Rightarrow \frac{1}{\|x_0\|} \ge \frac{1}{\|T^+\|\|b\|}.$$

Hence

$$\frac{\|y - x_0\|}{\|x_0\|} \ge \frac{\|\delta b\|}{\|T\|} \cdot \frac{1}{\|T^+\|\|b\|} = \frac{1}{\mathrm{cond}(T)} \cdot \frac{\|\delta b\|}{\|b\|}.$$

□

3. The stability of $Tx = b$ type equation with $(T + \Delta T)y = b$ type perturbation

We consider the following operator equations:

$$Tx = b \tag{3.1}$$

and

$$(T + \Delta T)y = b, \tag{3.2}$$

where $b \in R(T) \cap R(T + \Delta T), b \neq 0, \Delta T \in L(X, Y)$.

Theorem 3.1. *a) For each solution y of the equation* (3.2) *there exists a solution x_0 of the equation* (3.1) *such that*

$$\frac{\|y - x_0\|}{\|y\|} \leq \text{cond}(T) \cdot \frac{\|\Delta T\|}{\|T\|}.$$

b) If $\|\Delta T\| \cdot \|T^+\| < 1$ then, for each solution y of the ecuation (3.2) *there exists a solution x_0 for the equation* (3.1) *such that*

$$\frac{\|y - x_0\|}{\|x_0\|} \leq \text{cond}(T) \cdot \frac{\|\Delta T\|}{\|T\|} \cdot \frac{1}{1 - \|\Delta T\| \cdot \|T^+\|} =$$

$$= \text{cond}(T) \cdot \frac{\|\Delta T\|}{\|T\|} \left(1 + O(\Delta T)\right).$$

Proof. a) Let $x \in X$ be a solution of the equation (3.1) and $y \in X$ be a solution of the equation (3.2). Then $T(y - x) = -\Delta T y$ and consequently $\Delta T y \in R(T)$.

Let us consider $y \in X$ which verifies the equation $(T + \Delta T)y = b$ and $z \in N(T)$. We take $x := y + T^+ \Delta T - z$. Then $Tx = Ty + TT^+ \Delta T y - Tz = (T + \Delta T)y = b$, so x verifies the equation $Tx = b$.

Consider $x_0 := y + T^+ \Delta T y$ (which is the value corresponding to $z = 0$). Then $y - x_0 = T^+ \delta b \Rightarrow \|y - x_0\| \leq \|T^+\| \|\delta b\|$. Therefore

$$\frac{\|y - x_0\|}{\|y\|} \leq \frac{\|\Delta T\| \cdot \|T^+\| \cdot \|y\|}{\|y\|} = \text{cond}(T) \cdot \frac{\|\Delta T\|}{\|T\|}.$$

b) If $\|\Delta T\| \cdot \|T^+\| < 1$, then $\|\Delta T T^+\| < 1$. It follows that there exists $(\mathbf{I}_Y + \Delta T T^+)^{-1}$ and

$$\|(\mathbf{I}_Y + \Delta T T^+)^{-1}\| \leq \frac{1}{1 - \|\Delta T T^+\|} \leq \frac{1}{1 - \|\Delta T\| \cdot \|T^+\|}.$$

Let $x \in X$ be a solution of the equation (3.1) and $y \in X$ be a solution of the equation (3.2). From a) we have $T(y - x) = -\Delta T y$. It results that there exists $z \in N(T)$ such that $y - x = -T^+ \Delta T y + z$. Then $(\mathbf{I}_X + T^+ \Delta T)y = x + z$

and $y = (\mathbf{I}_X + T^+\Delta T)^{-1}(x+z)$. If we take x_0 corresponding to $z=0$, we obtain $y = (\mathbf{I}_X + T^+\Delta T)^{-1}x_0$ and $y - x_0 = -T^+\Delta T(\mathbf{I}_X + T^+\Delta T)^{-1}x_0$. Hence

$$\frac{\|y-x_0\|}{\|x_0\|} \leq \frac{\|T^+\Delta T\| \cdot \|\mathbf{I}_Y + \Delta T T^+\| \cdot \|x_0\|}{\|x_0\|} \leq$$

$$\leq \frac{\|\Delta T\| \cdot \|T^+\|}{1 - \|\Delta T\| \cdot \|T^+\|} = \operatorname{cond}(T) \cdot \frac{\|\Delta T\|}{\|T\|} \frac{1}{1 - \|\Delta T\| \cdot \|T^+\|} =$$

$$= \operatorname{cond}(T) \cdot \frac{\|\Delta T\|}{\|T\|} \left(1 + \frac{\|T^+\|}{\frac{1}{\|\Delta T\|} - \|T^+\|} \right) = \operatorname{cond}(T) \cdot \frac{\|\Delta T\|}{\|T\|} \left(1 + O(\Delta T)\right),$$

because, if $\|\Delta T\| \to 0$, then $\dfrac{\|T^+\|}{\frac{1}{\|\Delta T\|} - \|T^+\|} \to 0$. □

Theorem 3.2. *For each solution x of the equation* (3.1) *and each solution y of the equation* (3.2)*, we have*

$$\frac{\|y\|}{d(x, N(T))} \geq \frac{1}{\operatorname{cond}(T)} \cdot \frac{1}{1 + \frac{\|\Delta T\|}{\|T\|}} = \frac{1}{\operatorname{cond}(T)} \left(1 - O(\Delta T)\right).$$

Proof. From $(T + \Delta T)y = b$ it results that

$$\|y\| \geq \frac{\|b\|}{\|T + \Delta T\|}.$$

Since $Tx = b$ we infer that there exists $z \in N(T)$ such that $x = T^+b + z$. Then

$$\|x - z\| \leq \|T^+\| \cdot \|b\| \Rightarrow \frac{1}{\|x-z\|} \geq \frac{1}{\|T^+\| \cdot \|b\|},$$

and

$$\frac{\|y\|}{\|x-z\|} \geq \frac{1}{\|T^+\| \cdot \|T + \Delta T\|}.$$

Thus

$$\frac{\|y\|}{d(x, N(T))} \geq \frac{\|y\|}{\|x-z\|} \geq \frac{1}{\operatorname{cond}(T)} \cdot \frac{\|T\|}{\|T + \Delta T\|} \geq \frac{1}{\operatorname{cond}(T)} \cdot \frac{\|T\|}{\|T\| + \|\Delta T\|} =$$

$$= \frac{1}{\operatorname{cond}(T)} \cdot \frac{1}{1 + \frac{\|\Delta T\|}{\|T\|}} = \frac{1}{\operatorname{cond}(T)} \cdot \left(1 - \frac{1}{1 + \frac{\|T\|}{\|\Delta T\|}} \right) =$$

$$= \frac{1}{\operatorname{cond}(T)} \left(1 - O(\Delta T)\right),$$

because, if $\|\Delta T\| \to 0$, then $\dfrac{1}{1+\dfrac{\|T\|}{\|\Delta T\|}} \to 0.$ □

4. The stability of $Tx = b$ type equation with $(T+\Delta T)y = b + \Delta b$ type perturbation

We consider the following operator equations:

$$Tx = b \tag{4.1}$$

and

$$(T+\Delta T)y = b+\Delta b \tag{4.2}$$

where $b \in R(T), b+\Delta b \in R(T+\Delta T)$.

Theorem 4.1. *If $\|T^+\Delta T\| < 1$, then for each solution y of the equation* (4.2) *there exists a solution x_0 of the equation* (4.1) *such that*

$$\frac{\|y-x_0\|}{\|x_0\|} \le \frac{\mathrm{cond}(T)}{1-\|T^+\Delta T\|}\cdot\left(\frac{\|\Delta b\|}{\|b\|}+\frac{\|\Delta T\|}{\|T\|}\right)$$

Proof. Let y be a solution of the equation (4.2). From $Tx = b$ it results that there exists $z \in N(T)$ such that $x = T^+b + z$. From $T = TT^+T$, it follows that $z_0 = (\mathrm{I}_X - T^+T)y \in N(T)$. We denote $x_0 = T^+b + z_0$. Then $y - x_0 = T^+Ty - T^+b \in R(T^+) = R(T^*)$, so $y - x_0 = T^+T(y - x_0)$.

From (4.1) and (4.2) it results that $T(y-x_0) = \Delta b - \Delta Ty$. Thus $y-x_0 = T^+T(y-x_0) = T^+\Delta b - T^+\Delta Ty \Rightarrow y-x_0 = T^+\Delta b - T^+\Delta T(y-x_0) - T^+\Delta Tx_0 \Rightarrow$

$$(\mathbf{I}_X + T^+\Delta T)(y-x_0) = T^+(\Delta b - \Delta Tx_0).$$

From the hypothesis, there exists $(\mathbf{I}_X + T^+\Delta T)^{-1}$ and

$$\|(I+T^+\Delta T)^{-1}\| < \frac{1}{1-\|T^+\Delta T\|}.$$

Then

$$\frac{\|y-x_0\|}{\|x_0\|} \le \frac{1}{1-\|T^+\Delta T\|}\|T^+\|\cdot\frac{\|\Delta b - \Delta Tx_0\|}{\|x_0\|} \le$$

$$\frac{\mathrm{cond}(T)}{1-\|T^+\Delta T\|}\frac{\|\Delta b\|+\|\Delta Tx_0\|}{\|T\|\cdot\|x_0\|} \le \frac{\mathrm{cond}(T)}{1-\|T^+\Delta T\|}\cdot\left(\frac{\|\Delta b\|}{\|b\|}+\frac{\|\Delta T\|}{\|T\|}\right).$$

□

5. The stability of $\|Tx - b\| = \inf_{z\in X} \|Tz - b\|$ type equation with $\|Ty - b - \delta b\| = \inf_{z\in X} \|Tz - b - \delta b\|$ type perturbation

We consider the following operator equations:

$$\|Tx - b\| = \inf_{z\in X} \|Tz - b\| \tag{5.1}$$

and

$$\|Ty - b - \delta b\| = \inf_{z\in X} \|Tz - b - \delta b\| \tag{5.2}$$

with $b, b + \delta b \in Y, b \notin R(T)^{\perp}$.

Theorem 5.1. *a) For each solution x of the equation* (5.1) *there exists a solution y_0 of the equation* (5.2) *such that*

$$\frac{\|y_0 - x\|}{\mathrm{dist}(x, N(T))} \le \mathrm{cond}(T) \cdot \frac{\|\delta b\|}{\|TT^{+}b\|}.$$

b) There exists a solution x_0 of the equation (5.1) *and there exists a solution y_0 of the equation* (5.2) *such that*

$$\frac{\|y_0 - x_0\|}{\|x_0\|} \le \mathrm{cond}(T) \cdot \frac{\|\delta b\|}{\|TT^{+}b\|}.$$

Proof. a) Let $x \in X$ be a solution of the equation (5.1). Then $x = T^{+}b + z$, where $z \in N(T)$. Let $y = x + T^{+}\delta b + z'$, where $z' \in N(T)$. Then

$$\|Ty - b - \delta b\| = \|Tx + TT^{+}\delta b - b - \delta b\| =$$

$$= \|TT^{+}(b + \delta b) - b - \delta b\| = \inf_{z\in X} \|Tz - b - \delta b\|,$$

so y is a solution of equation (5.2). We consider $y_0 = x + T^{+}\delta b$. It results that

$$\|y_0 - x\| \le \|T^{+}\| \cdot \|\delta b\|.$$

Since

$$\|x - z\| = \|T^{+}b\|,$$

we deduce that

$$\frac{\|y_0 - x\|}{\|x - z\|} \le \frac{\|T^{+}\| \cdot \|\delta b\|}{\|T^{+}b\|} = \mathrm{cond}(T) \cdot \frac{\|\delta b\|}{\|T\| \cdot \|T^{+}b\|} \le \mathrm{cond}(T) \cdot \frac{\|\delta b\|}{\|TT^{+}b\|}.$$

Then

$$\frac{\|y_0 - x\|}{\mathrm{dist}(x, N(T))} \le \frac{\|y_0 - x\|}{\|x - z\|} \le \mathrm{cond}(T) \cdot \frac{\|\delta b\|}{\|TT^{+}b\|}.$$

b) We take $x_0 = T^+b$ and $y_0 = x_0 + T^+\delta b$. Then

$$\|y_0 - x_0\| \leq \|T^+\| \cdot \|\delta b\| \text{ and } \|x_0\| = \|T^+b\|,$$

so

$$\frac{\|y_0 - x_0\|}{\|x_0\|} \leq \frac{\|T^+\| \cdot \|\delta b\|}{\|T^+b\|} = \operatorname{cond}(T) \cdot \frac{\|\delta b\|}{\|T\| \cdot \|T^+b\|} \leq \operatorname{cond}(T) \cdot \frac{\|\delta b\|}{\|TT^+b\|}.$$

□

Theorem 5.2. *a) For any solution x of the equation* (5.1) *and for any solution y of the equation* (5.2) *we have*

$$\frac{\|y - x\|}{\operatorname{dist}(x, N(T))} \geq \frac{1}{\operatorname{cond}(T)} \cdot \frac{\|TT^+\delta b\|}{\|b\|}.$$

b) For any solution y of the equation (5.2) *there exists a solution x_0 of the solution* (5.1) *such that*

$$\frac{\|y - x_0\|}{\|x_0\|} \geq \frac{1}{\operatorname{cond}(T)} \cdot \frac{\|TT^+\delta b\|}{\|b\|}.$$

Proof. Let $x \in X$ be a solution of the equation (5.1) and $y \in X$ a solution of the equation (5.2). Then $x = T^+b + z$, with $z \in N(T)$ and $y = T^+b + T^+\delta b + z' = x + T^+\delta b + z'$, with $z' \in N(T)$. Then $T(y-x) = TT^+\delta b$ and

$$\|y - x\| \geq \frac{\|TT^+\delta b\|}{\|T\|}.$$

a) For $z \in N(T)$ we have

$$\|x - z\| = \|T^+b\| \leq \|T^+\|\|b\| \Rightarrow \frac{1}{\|x - z\|} \geq \frac{1}{\|T^+\|\|b\|} \Rightarrow$$

$$\frac{1}{\operatorname{dist}(x, N(T))} \geq \frac{1}{\|T^+\|\|b\|}.$$

Therefore

$$\frac{\|y - x\|}{\operatorname{dist}(x, N(T))} \geq \frac{\|TT^=\delta b\|}{\|T\|} \cdot \frac{1}{\|T^+\|\|b\|} = \frac{1}{\operatorname{cond}(T)} \cdot \frac{\|TT^+\delta b\|}{\|b\|}.$$

b) For $x_0 = T^+b$ we have

$$\|x_0\| = \|T^+b\| \leq \|T^+\|\|b\| \Rightarrow \frac{1}{\|x_0\|} \geq \frac{1}{\|T^+\|\|b\|}.$$

Hence

$$\frac{\|y - x_0\|}{\|x_0\|} \geq \frac{\|TT^+\delta b\|}{\|T\|} \cdot \frac{1}{\|T^+\|\|b\|} = \frac{1}{\operatorname{cond}(T)} \cdot \frac{\|TT^+\delta b\|}{\|b\|}.$$

□

6. The stability of $\|Tx-b\| = \inf_{z\in X}\|Tz-b\|$ type equation with $\|(T+\Delta T)y-b-\delta b\| = \inf_{z\in X}\|(T+\Delta T)z-b-\delta b\|$ type perturbation

We consider the equations

$$\|Tx-b\| = \inf_{z\in X}\|Tz-b\| \tag{6.1}$$

and

$$\|(T+\Delta T)y-b-\delta b\| = \inf_{z\in X}\|(T+\Delta T)z-b-\delta b\| \tag{6.2}$$

where $b\in Y$.

In this case, $T+\Delta T$ may fail to have closed range and then $(T+\Delta T)^+$ may not exist. We chose for study a particular case. Thus, we suppose that

$$\left\|T^+\right\|\|\Delta T\| < 1,$$

$$R(T+\Delta T) = R(T) \text{ and } N(T+\Delta T) = N(T).$$

Then there exist the operators $(I+T^+\Delta T)^{-1}$, $(I+\Delta TT^+)^{-1}$ and

$$(T+\Delta T)^+ = (I+T^+\Delta T)^{-1}T^+ = T^+(I+\Delta TT^+)^{-1}.$$

Theorem 6.1. *For each solution y of the equation* (6.2) *there exists a solution x_0 of the equation* (6.1) *such that*

$$\frac{\|y-x_0\|}{\|x_0\|} \le \frac{\operatorname{cond}(T)}{1-\|T^+\Delta T\|}\cdot\left(\frac{\|\delta b\|}{\|TT^+b\|}+\frac{\|\Delta T\|}{\|T\|}\right).$$

Proof. Let y be a solution of the equation (6.2). Then $y=(T+\Delta T)^+(b+\delta b)+z$, where $z\in N(T+\Delta T)=N(T)$. Because $T=TT^+T$, it results that $z_0=(I-T^+T)y\in N(T)$. We denote by $x_0=T^+b+z_0$. Then x_0 is a solution of the equation (6.1) and $y-x_0=T^+Ty-T^+b\in R(T^+)=R(T^*)$. Therefore $y-x_0=T^+T(y-x_0)$.

We have $y-x_0=(T+\Delta T)^+(b+\delta b)-T^+b+z-z_0$. Since $(T+\Delta T)^+ = T^+(I+\Delta TT^+)^{-1}$ it results that $T(y-x_0) = TT^+(I+\Delta TT^+)^{-1}(b+\delta b)-TT^+b \Rightarrow T^+T(y-x_0) = T^+(I+\Delta TT^+)^{-1}(b+\delta b)-T^+b = (I+T^+\Delta T)^{-1}T^+(b+\delta b)-T^+b$. Then

$$(I+T^+\Delta T)(y-x_0) = T^+(b+\delta b)-(I+T^+\Delta T)T^+b =$$

$$= T^+(\delta b-\Delta TT^+b) = T^+(\delta b-\Delta Tx_0).$$

From the hypothesis there exists $(I+T^+\Delta T)^{-1}$ and

$$\|(I+T^+\Delta T)^{-1}\| < \frac{1}{1-\|T^+\Delta T\|}.$$

Then
$$\frac{\|y-x_0\|}{\|x_0\|} \leq \frac{1}{1-\|T^+\Delta T\|}\|T^+\| \cdot \frac{\|\delta b - \Delta T x_0\|}{\|x_0\|} \leq$$
$$\frac{\mathrm{cond}(T)}{1-\|T^+\Delta T\|}\frac{\|\delta b\| + \|\Delta T x_0\|}{\|T\| \cdot \|x_0\|} \leq \frac{\mathrm{cond}(T)}{1-\|T^+\Delta T\|} \cdot \left(\frac{\|\delta b\|}{\|T x_0\|} + \frac{\|\Delta T\|}{\|T\|}\right) \leq$$
$$\leq \frac{\mathrm{cond}(T)}{1-\|T^+\Delta T\|} \cdot \left(\frac{\|\delta b\|}{\|TT^+ b\|} + \frac{\|\Delta T\|}{\|T\|}\right).$$
□

References

[1] A. BEN-ISRAEL and T.N.E. GREVILLE, *Generalized Inverses*, Springer-Verlag, 2003.

[2] J. DING and L.J. HUANG, Perturbation of generalized inverses of linear operators in Hilbert spaces, *J. Math. Anal. Appl.*, **198** (1996), 506-515.

[3] J. DING and Y. WEI, Bounds for perturbed solutions of linear operator equations in Hilbert space, *Appl. Math. Comput.*, **132** (2002), 293-298.

[4] G. CHEN and Y. XUE, The expression of the generalized inverse of the perturbed operator under Type I perturbation in Hilbert spaces, *Linear Algebra Appl.*, **285** (1998), 1-6.

Daniel Stănică
University of Bucharest, Faculty of Mathematics and Computer Science
14 Academiei Street, 010014 Bucharest, Romania
E-mail: stanicad@fmi.unibuc.ro