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Clifford structures on a Hilbert space H and bases for module structures on $\mathcal{B}(H)$

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Abstract - Let H be a complex Hilbert space, $\mathcal{R}$ a unital ring associated to a $\mathcal{R}$-module structure on the algebra $\mathcal{B}(H)$ of all bounded linear operators on H and R a Clifford structure (of real dimension 2) on H. We describe some properties which connect Clifford structure R with the $\mathcal{R}$-module structure in order to obtain $\{I, R(1), R(\mathrm{i}), \mathrm{i}R(1)R(\mathrm{i})\}$ a basis of the $\mathcal{R}$-module structure on $\mathcal{B}(H)$.

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Introduction

In the following $\mathbb{C}$ is the field of complex numbers, $\mathcal{H}$ will be a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the C^*- algebra of all linear bounded operators on $\mathcal{H}$ and $H = \mathcal{H} \oplus \mathcal{H}$ the Hilbert sum of the pair $(\mathcal{H}, \mathcal{H})$. In order to describe the spectrum for a normal relation in $\mathcal{H}$, J. Ph. Labrousse used in [2] the following $\mathcal{B}(H)$-module basis in $\mathcal{B}(H) = \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$:

$$I = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I} \end{pmatrix}, \; J = \begin{pmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{pmatrix}, \; K = \begin{pmatrix} 0 & -\mathrm{i}\mathcal{I} \\ \mathrm{i}\mathcal{I} & 0 \end{pmatrix}, \; L = \begin{pmatrix} -\mathcal{I} & 0 \\ 0 & \mathcal{I} \end{pmatrix}, \qquad (0)$$

where $\mathcal{I}$ is the identity of $\mathcal{H}$.

The aim of the paper is to show that the algebraic properties obtained from the above description (which are used in [2] in order to associate the spectrum of a normal relation in $\mathcal{H}$ with the Taylor-joint spectrum of a 3-tuple of bounded operators on $\mathcal{H}$) can be generally derived from a class of Clifford structures R (of real dimension 2) on an arbitrary Hilbert space H when R is connected in some way with a $\mathcal{R}$-module structure on $\mathcal{B}(H)$ given by an unital ring $\mathcal{R}$.

1. Clifford structures on H and $\mathcal{R}$-module structure on $\mathcal{B}(H)$

First we recall that a *Clifford structure* (of real dimension 2) on a complex Hilbert space H is a real linear mapping $R : \mathbb{C} \to \mathcal{B}(H)$ into the space of self adjoint operators on H that satisfies

$$R(z)^2 = |z|^2 I,\ I \text{ being the identity operator on } H,\ z \in \mathbb{C}. \tag{1.1}$$

Corresponding to (1.1) one finds the multiplication tables

	$R(1)$	$R(\mathrm{i})$	$R(1)R(\mathrm{i})$
$R(1)$	I	$R(1)R(\mathrm{i})$	$R(\mathrm{i})$
$R(\mathrm{i})$	$-R(1)R(\mathrm{i})$	I	$-R(1)$
$R(1)R(\mathrm{i})$	$-R(\mathrm{i})$	$R(1)$	$-I$

(1.2)

	$R(1)$	$R(\mathrm{i})$	$\mathrm{i}R(1)R(\mathrm{i})$
$R(1)$	I	$R(1)R(\mathrm{i})$	$\mathrm{i}R(\mathrm{i})$
$R(\mathrm{i})$	$-R(1)R(\mathrm{i})$	I	$-\mathrm{i}R(1)$
$\mathrm{i}R(1)R(\mathrm{i})$	$-\mathrm{i}R(\mathrm{i})$	$\mathrm{i}R(1)$	I

We obviously have

$$\begin{aligned} \mathrm{alg}_{\mathbb{C}}\{\mathrm{range}\, R\} &= \mathrm{sp}_{\mathbb{C}}\{I, R(1), R(\mathrm{i}), \mathrm{i}R(1)R(\mathrm{i})\} \\ &= \mathrm{sp}_{\mathbb{C}}\{I, R(1), R(\mathrm{i}), R(1)R(\mathrm{i})\}. \end{aligned}$$

Remark 1.1. If $H = \mathcal{H} \oplus \mathcal{H}$ and I, J, K, L are the operator matrices in $H = \mathcal{H} \oplus \mathcal{H}$ as in (0), then putting $R(1) = J$, $R(\mathrm{i}) = K$ and $R(\alpha + \mathrm{i}\beta) = \alpha R(1) + \beta R(\mathrm{i}) = \alpha J + \beta K$ we obtain a Clifford structure on H based on the multiplication table

	J	K	L
J	I	$-\mathrm{i}L$	$\mathrm{i}K$
K	$\mathrm{i}L$	I	$-\mathrm{i}J$
L	$-\mathrm{i}K$	$\mathrm{i}J$	I

Indeed, for $z = \alpha + \mathrm{i}\beta$,

$$R(z)^2 = (\alpha R(1) + \beta R(\mathrm{i}))^2 = \left(\alpha^2 + \beta^2\right) I + \alpha\beta JK + \alpha\beta KJ = |z|^2\, I.$$

We can also observe that $\mathrm{i}JK = L$ as in (1.2) for $R(1) = J$, $R(\mathrm{i}) = K$.

Obviously in this case, the $\mathcal{B}(\mathcal{H})$-module basis $\{I, J, K, L\}$ of $\mathcal{B}(H)$ can be rewritten (using Clifford structure R given by $R(1) = J$, $R(\mathrm{i}) = K$) as $\{I, R(1), R(\mathrm{i}), \mathrm{i}R(1)R(\mathrm{i})\}$.

Now let H be a Hilbert space with $\mathcal{B}(H)$ having a $\mathcal{R}$-module structure given by an unital ring $\mathcal{R}$. The following definition describes a class of Clifford structures on H related to this $\mathcal{R}$-module structure in a way corresponding to our purpose announced in Introduction.

Definition 1.1. *Let R be a Clifford structure on H and $J = R(1)$, $K = R(\mathrm{i})$, $L = \mathrm{i}R(1)R(\mathrm{i})$ as in (1.2). We say that the Clifford structure R on H and a $\mathcal{R}$-module structure on $\mathcal{B}(H)$ are connected if for every $r \in \mathcal{R}$ the following two properties hold:*

(c_1) $rI \in \{J, K, L\}'$;

(c_2) $(rM_1)\,M_2 = \begin{cases} -M_2\,(rM_1) & \text{if} \quad M_1 \neq M_2 \\ \;\;\, M_1\,(rM_1) & \text{if} \quad M_1 = M_2 \end{cases}$ *for* $M_1, M_2 \in \{J, K, L\}$.

Example 1.1. (e_1) Let us consider $H = \mathcal{H} \oplus \mathcal{H}$, $\mathcal{R} = B(\mathcal{H})$, the canonical $\mathcal{R}$-module structure on $\mathcal{B}(H)$ given by $r\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$ for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{B}(H)$, where a, b, c, d, r are contained in $\mathcal{B}(\mathcal{H})$ and the Clifford structure given by

$$R(1) = J = \begin{pmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{pmatrix},\ R(\mathrm{i}) = K = \begin{pmatrix} 0 & -\mathrm{i}\mathcal{I} \\ \mathrm{i}\mathcal{I} & 0 \end{pmatrix},\ \mathrm{i}JK = L = \begin{pmatrix} -\mathcal{I} & 0 \\ 0 & \mathcal{I} \end{pmatrix}$$

Let us prove that the above Clifford structure and the $\mathcal{R}$-module structure on $\mathcal{B}(H)$ are connected. Firstly we can write,

$$\begin{aligned} (rU_1)\,U_2 &= r\,(U_1U_2) \text{ for every } r \in \mathcal{R} \text{ and } U_1, U_2 \in \mathcal{B}(H) \\ (rU_1)\,U_2 &= U_1\,(rU_2) \text{ for every } U_1, U_2 \in \mathcal{B}(H) \text{ and } r \in \{a_1, b_1, c_1, d_1\}' \end{aligned}$$

if

$$U_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}. \tag{1.3}$$

So, the following implication is true:

$$r \in \{a_1, b_1, c_1, d_1\}' \text{ and } [U_1, U_2] = 0 \Rightarrow [rU_2, U_1] = 0. \tag{1.4}$$

Indeed, we have $(rU_2)\,U_1 = r\,(U_2U_1) = r\,(U_1U_2) = (rU_1)\,U_2 = U_1rU_2$. If $U_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \{J, K, L\}$, then $r \in \{a_1, b_1, c_1, d_1\}'$ for every $r \in \mathcal{R} = B(\mathcal{H})$ and $U_2 = I$ obviously verifies $[U_1, U_2] = 0$. So, by the above implication (1.4), we deduce $[rI, U] = 0$ for every $U \in \{J, K, L\}$ i.e. $rI \in \{J, K, L\}'$, hence (c_1) holds.

For $M_1, M_2 \in \{J, K, L\}$ we have

$$r\,(M_1M_2) = (rM_1)\,M_2 = M_1\,(rM_2) \tag{1.5}$$

because for $M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \{J, K, L\}$ we have $r \in \{a_1, b_1, c_1, d_1\}'$ for every $r \in \mathcal{R} = B(\mathcal{H})$.

For $M_1 = M_2 \in \{J, K, L\}$ we deduce by (1.5), $(rM_1)\, M_1 = M_1\,(rM_1)$.

If $M_1 \neq M_2$ and $M_1, M_2 \in \{J, K, L\}$, then $M_1M_2 = -M_2M_1$. Then we have

$$r\,(M_1M_2) = r\,(-M_2M_1) = -r\,(M_2M_1)\,.$$

So, by (1.5) we deduce $(rM_1)\,M_2 = -\,(rM_2)\,M_1$ and (c_2) is completely proved.

(e_2) Let H be a complex Hilbert space, $R : \mathbb{C} \to \mathcal{B}(H)$ a Clifford structure, I the identity on H, $J = R(1)$, $K = R(\mathrm{i})$, $L = \mathrm{i}R(1)R(\mathrm{i})$. Let us consider also the unital ring $\mathcal{R} \subset \{J, K\}' = (\mathrm{alg}_{\mathbf{C}}\{I, J, K, L\})'$, where $\{\ \}'$ denotes the commutant in $\mathcal{B}(H)$ and the $\mathcal{R}$-module structure on $\mathcal{B}(H)$ is given by multiplication in $\mathcal{B}(H)$.

The Clifford structure and this $\mathcal{R}$-module structure are obviously connected (Definition 1.1).

Remark 1.2. Let us consider the example (e_1). It is well known that $\{I, J, K, L\}$ is a $\mathcal{B}(\mathcal{H})$-module basis of $\mathcal{B}(H)$. We also recall that $R(\alpha+\mathrm{i}\beta) = \alpha J+\beta K$ defines a Clifford structure and $R(1) = J$, $R(\mathrm{i}) = K$, $\mathrm{i}R(1)R(\mathrm{i}) = L$. We will prove that this property is also true for an arbitrary Hilbert space H and some $\mathcal{R}$-module structure on $\mathcal{B}(H)$ which are connected with a Clifford structure (of real dimension 2) on H as in Definition 1.1.

2. Bases of $\mathcal{R}$-module structures connected with Clifford structure on $\mathcal{B}(H)$

Let H be a complex Hilbert space, $\mathcal{R}$ an unital ring and a $\mathcal{R}$-module structure on $\mathcal{B}(H)$ connected with a Clifford structure R (of real dimension 2) on H as in Definition 1.1. We recall the notations: I for the identity operator on H, $J = R(1)$, $K = R(\mathrm{i})$, $L = \mathrm{i}R(1)R(\mathrm{i})$.

Proposition 2.1. *Let $\{I, J, K, L\} \subset \mathcal{B}(H)$ be the system corresponding to R, the Clifford structure (of real dimension 2) connected with a $\mathcal{R}$-module structure on $\mathcal{B}(H)$. If $U = u_0I + u_1J + u_2K + u_3L$ is an element of the $\mathcal{R}$-module generated in $\mathcal{B}(H)$ by $\{I, J, K, L\}$ with $u_0, u_1, u_2, u_3 \in \mathcal{R}$, then u_0I, u_1J, u_2K, u_3L are uniquely given by U.*

Proof. If $U = u_0I+u_1J+u_2K+u_3L$, we can perform a direct computation by using $(c_1), (c_2), (1.2)$ and we obtain as in [2]

$$\begin{aligned} JUJ &= u_0I + u_1J - u_2K - u_3L \\ KUK &= u_0I - u_1J + u_2K - u_3L \\ LUL &= u_0I - u_1J - u_2K + u_3L. \end{aligned} \tag{2.1}$$

So we have

$$\begin{aligned} u_0I &= \tfrac{1}{4}\left(U + JUJ + KUK + LUL\right)\\ u_1J &= \tfrac{1}{4}\left(U + JUJ - KUK - LUL\right)\\ u_2K &= \tfrac{1}{4}\left(U - JUJ + KUK - LUL\right)\\ u_3L &= \tfrac{1}{4}\left(U - JUJ - KUK + LUL\right), \end{aligned} \tag{2.2}$$

which completes the proof. □

Definition 2.1. *Consider a $\mathcal{R}$-module structure on $\mathcal{B}(H)$ connected with a Clifford structure R (of real dimension 2). The $\mathcal{R}$-module structure is called I-faithful if the following conditions are satisfied:*

(f_1) $r \in \mathcal{R}$, $rI = 0 \Rightarrow r = 0$

(f_2) $(rI)M = rM$ *for every* $M \in \{J, K, L\}$ *and* $r \in \mathcal{R}$.

Proposition 2.2. *Let us consider a $\mathcal{R}$-module structure on $\mathcal{B}(H)$ connected with a Clifford structure R (of real dimension 2) on H and I-faithful. Then $\{I, J, K, L\}$, the system of operators associated to R, is a $\mathcal{R}$-modul basis in the $\mathcal{R}$-module generated by $\{I, J, K, L\}$.*

Proof. We have to prove that $u_0, u_1, u_2, u_3 \in \mathcal{R}$ are uniquely determined by the property $U = u_0I + u_1J + u_2K + u_3L$ for a fixed $U \in \mathcal{B}(H)$. By the above proposition (the equalities (2.2)) u_0I, u_1J, u_2K, u_3L are uniquely given by U. By Definition 2.1 we deduce

$$(rM)M = (rI)MM = (rI)M^2 = rM^2 = rI, \text{ for all } r \in \mathcal{R},\ M \in \{J, K, L\}.$$

Multiplying the equalities (2.2) respectively by I, J, K, L we obtain, by the above remark,

$$\begin{aligned} u_0I &= \tfrac{1}{4}\left(U + JUJ + KUK + LUL\right)\\ u_1I &= \tfrac{1}{4}\left(UJ + JUJ - \mathrm{i}(KUL - LUK)\right)\\ u_2I &= \tfrac{1}{4}\left(UK + KU - \mathrm{i}(LUJ - JUL)\right)\\ u_3I &= \tfrac{1}{4}\left(UL + LU - \mathrm{i}(JUK - KUJ)\right). \end{aligned} \tag{2.3}$$

So, by condition (f_1) in Definition 2.1, we obtain that u_0, u_1, u_2, u_3 are uniquely given by U. □

Now let H, $\mathcal{B}(H)$ be as above and J, K, L given by a Clifford structure of real dimension 2 on H by the usual equalities $J = R(1)$, $K = R(\mathrm{i})$, $L = \mathrm{i}R(1)R(\mathrm{i})$. We denote the linear mappings given by the second parts of the equalities (2.3) as follows:

$$\begin{aligned} &\widetilde{R_i}\colon \mathcal{B}(H) \to \mathcal{B}(H), \quad i = 0, 1, 2, 3,\\ &\widetilde{R_0}(U) = U + JUJ + KUK + LUL\\ &\widetilde{R_1}(U) = UJ + JU - \mathrm{i}(KUL - LUK)\\ &\widetilde{R_2}(U) = UK + KU - \mathrm{i}(LUJ - JUL)\\ &\widetilde{R_3}(U) = UL + LU - \mathrm{i}(JUK - KUJ), \end{aligned}$$

for every $U \in \mathcal{B}(H)$.

A simple computation gives the inclusion

$$\bigcup_{i=0}^{3} \text{range}\left(\widetilde{R_i}\right) \subset \{J, K\}' = \{I, J, K, L\}'. \tag{2.4}$$

Let us also consider a $\mathcal{R}$-module structure on $\mathcal{B}(H)$ and J, K, L given, as above, by an arbitrary fixed Clifford structure R (of real dimension 2) on H.

Proposition 2.3. *Assume the above setting. If for every $M \in \{J, K, L\}$ and $r \in \mathcal{R}$ we have $\bigcup_{i=0}^{3} \text{range}\left(\widetilde{R_i}\right) \subset \mathcal{R}I$ and $(rI)M = rM$, then the $\mathcal{R}$-module generated by $\{I, J, K, L\}$ is $\mathcal{B}(H)$ i.e. $\{I, J, K, L\}$ is a system of generators for $\mathcal{R}$-module structure on $\mathcal{B}(H)$.*

Proof. By the hypothesis, for every $U \in \mathcal{B}(H)$ there are $u_i \in \mathcal{R}$, so that,

$$u_i I = \frac{1}{4}\widetilde{R_i}(U), \qquad i = 0, 1, 2, 3.$$

On the other hand $(u_i I)M = u_i M$ for $i = 0, 1, 2, 3$ and $M \in \{J, K, L\}$. So, by the definition of $\widetilde{R_i}$ and u_i, we deduce:

$$\begin{aligned}
u_0 I &= \frac{1}{4}\widetilde{R_0}(U) = \frac{1}{4}(U + JUJ + KUK + LUL) \\
u_1 J &= (u_1 I)\, J = \frac{1}{4}\widetilde{R_1}(U)J = \frac{1}{4}(U + JUJ - KUK - LUL) \\
u_2 K &= (u_2 I)\, K = \frac{1}{4}\widetilde{R_2}(U)K = \frac{1}{4}(U - JUJ + KUK - LUL) \\
u_3 L &= (u_3 I)\, L = \frac{1}{4}\widetilde{R_3}(U)L = \frac{1}{4}(U - JUJ - KUK + LUL).
\end{aligned}$$

Summing up these equalities, it results that for every $U \in \mathcal{B}(H)$ there exist $u_0, u_1, u_2, u_3 \in \mathcal{R}$ such that

$$U = u_0 I + u_1 J + u_2 K + u_3 L,$$

which concludes the proof. □

By Propositions 2.2 and 2.3 we deduce the following theorem.

Theorem 2.1. *If H is a complex Hilbert space, R a Clifford structure of real dimension 2 on H and $\{I, J, K, L\}$ the canonical system associated to R, then $\{I, J, K, L\}$ is a basis of $\mathcal{B}(H)$ for every $\mathcal{R}$-module structure of $\mathcal{B}(H)$ connected with R, I-faithful and verifying*

$$\bigcup_{i=0}^{3} \text{range}\widetilde{R_i} \subset \mathcal{R}I.$$

Proof. We can apply Propositions 2.2 and 2.3. □

Example 2.1. The example (e_1) verifies the hypothesis of the above Theorem 2.1. The example (e_2) verifies the hypothesis of the above Theorem 2.1 if and only if the unital subring $\mathcal{R}$ of $\mathcal{B}(H)$ verifies $\bigcup_{i=0}^{3} \text{range}\widetilde{R_i} \subset \mathcal{R} \subset \{I, J, K, L\}'$ and the existence of such a $\mathcal{R}$ is an easy consequence of (2.4) (the extremal cases being the unital ring generated by $\bigcup_{i=0}^{3} \text{range}\widetilde{R_i}$ or $\{I, J, K, L\}'$). So the conclusion of the above Theorem holds in these two cases.

Indeed, it was proved that in both examples (e_1) and (e_2) the $\mathcal{R}$-module structure and Clifford structures which are considered are connected. Since in both examples the $\mathcal{R}$-module multiplication is given by the multiplication in $\mathcal{B}(H)$, it follows that the $\mathcal{R}$-module structures under consideration are I-faithful. The last property $\bigcup_{i=0}^{3} \text{range}\widetilde{R_i} \subset \mathcal{R}I$ (the last condition in the in the hypothesis of the Theorem 2.1) is in both examples (e_1) and (e_2), the inclusion $\bigcup_{i=0}^{3} \text{range}\widetilde{R_i} \subset \mathcal{R}$ because $\mathcal{R} \subset \mathcal{B}(H)$ and I is the identity on H. In the first case, i.e. example (e_1), $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ and the inclusion is obvious. In the second case we considered example (e_2) with the property $\bigcup_{i=0}^{3} \text{range}\widetilde{R_i} \subset \mathcal{R} \subset \{I, J, K, L\}'$ and the inclusion is also obvious.

So we have:

(a_1) For $H = \mathcal{H} \oplus \mathcal{H}$, $\mathcal{H}$ complex Hilbert space and $I = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I} \end{pmatrix}$, $J = \begin{pmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & -\mathrm{i}\mathcal{I} \\ \mathrm{i}\mathcal{I} & 0 \end{pmatrix}$, $L = \begin{pmatrix} -\mathcal{I} & 0 \\ 0 & \mathcal{I} \end{pmatrix}$ the well known result (cf. [2]) that $\{I, J, K, L\}$ is a basis of $\mathcal{B}(\mathcal{H})$ -module structure of $\mathcal{B}(H)$, derived from a Clifford structure connected with the $\mathcal{B}(\mathcal{H})$-module structure.

(a_2) For an arbitrary complex Hilbert space H, every Clifford structure R of real dimension 2 on H, I the identity on H, $J = R(1)$, $K = R(\mathrm{i})$, $L = \mathrm{i}R(1)R(\mathrm{i})$ give $\{I, J, K, L\}$ a $\mathcal{R}$-module basis of $\mathcal{B}(H)$ for every unital subring $\mathcal{R}$ of $\mathcal{B}(H)$ verifying $\bigcup_{i=0}^{3} \text{range}\widetilde{R_i} \subset \mathcal{R} \subset \{I, J, K, L\}'$.

3. The composition of operators on a complex Hilbert space endowed with a Clifford structure

In this section we take a close look at the composition of two linear bounded operators on a complex Hilbert space H when on H is given a Clifford structure of real dimension 2 which is connected with some $\mathcal{R}$-module structure I-faithful on $\mathcal{B}(H)$. In the following H will be a complex Hilbert space, $\mathcal{B}(H)$ the algebra of all linear bounded operators on H and $R : \mathbb{C} \to \mathcal{B}(H)$ a Clifford structure of real dimension 2, I the identity on H and $J = R(1)$, $K = R(\mathrm{i})$, $L = \mathrm{i}R(1)R(\mathrm{i})$.

Let us consider a unital ring $\mathcal{R}$ and a $\mathcal{R}$-module structure connected with R, I-faithful on $\mathcal{B}(H)$ such that $\bigcup_{i=0}^{3} \mathrm{range}\widetilde{R_i} \subset \mathcal{R}I$.

If $U, V \in \mathcal{B}(H)$ and

$$U = u_0 I + u_1 J + u_2 K + u_3 L, \quad V = v_0 I + v_1 J + v_2 K + v_3 L,$$

where $u_i, v_i \in \mathcal{R}$, $i = 0, 1, 2, 3$ are the coordinates of U, V in the $\mathcal{R}$-module basis $\{I, J, K, L\}$ of $\mathcal{B}(H)$ (see the above Theorem 2.1), then the coordinates (w_i) of UV in the the same basis can be obtained by the action in $\mathcal{R}^4$ of some matrix of $\mathcal{M}_4(\mathcal{R})$. More precisely we obtain by computation (as in [2]):

$$\begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ u_1 & u_0 & \mathrm{i}u_3 & -\mathrm{i}u_2 \\ u_2 & -\mathrm{i}u_3 & u_0 & \mathrm{i}u_1 \\ u_3 & \mathrm{i}u_2 & -\mathrm{i}u_1 & u_0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

i.e. the multiplication by U in $\mathcal{B}(H)$ has in the $\mathcal{R}$-module basis $\{I, J, K, L\}$ of $\mathcal{B}(H)$, the following matrix $T(U) \in \mathcal{M}_4(\mathcal{R})$:

$$T(U) = \begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ u_1 & u_0 & \mathrm{i}u_3 & -\mathrm{i}u_2 \\ u_2 & -\mathrm{i}u_3 & u_0 & \mathrm{i}u_1 \\ u_3 & \mathrm{i}u_2 & -\mathrm{i}u_1 & u_0 \end{pmatrix}.$$

Remark 3.1. To determine $T(U)$ we compute UV using $\{I, J, K, L\}$ as basis and multiplication table (1.2) for I, J, K, L, using that the $\mathcal{R}$-module structure is connected with $\mathcal{R}$ and I-faithful.

Proposition 3.1. *Let us consider $T : \mathcal{B}(H) \to \mathcal{M}_4(\mathcal{R})$, $U \mapsto T(U)$, where $T(U)$ is the matrix in the $\mathcal{R}$-module basis $\{I, J, K, L\}$ of $\mathcal{B}(H)$ of the $\mathcal{R}$-linear operator M_U, the multiplication by U on $\mathcal{B}(H)$. The map T is $\mathcal{R}$-linear and multiplicative.*

Proof. We have

$$\begin{aligned} M_{UV} &= M_U M_V \text{ i.e. } T(UV) = T(U)T(V) \\ M_{U+V} &= M_U + M_V \text{ i.e. } T(U+V) = T(U) + T(V) \\ M_{rU} &= M_{rIU} = M_{rI} M_U \text{ i.e. } T(rU) = T(rI)T(U) \\ &= \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \end{pmatrix} \cdot T(U) = rT(U). \end{aligned}$$

This concludes the proof. □

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References

[1] W. ARVERSON, The Dirac operator of a commuting d-tuple, *J. Funct. Anal.*, **189** (2002), 53-79.

[2] J.-PH. LABROUSSE, *The joint spectrum associated to a closed linear relation*, preprint, 2009.

[3] J.-PH. LABROUSSE, *Idempotent linear relations*, Spectral Analysis and Its Applications, Theta Ser. Adv. Math., vol. 2, pp. 129-149, Theta, Bucharest, 2003.

[4] J.-PH. LABROUSSE, Les opérateurs quasi Fredholm: une généralisation des opérateurs semi Fredholm, *Rend. Circ. Mat. Palermo (2)*, **29** (1980), 161-258.

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