Annals of the University of Bucharest (mathematical series)
(Analele Universităţii Bucureşti. Matematică)
1 (LIX) (2010), 145–154

The evolution of Erret Bishop's ideas in abstract spectral theory[1]

Mihai Putinar

Communicated by Viorel Iftimie

To Professor Ion Colojoară on the occasion of his 80th birthday, with admiration and best wishes for the next decades

Abstract - A survey of E. Bishop's PhD dissertation is linked to advances in abstract spectral decompositions, until 1988.

Key words and phrases : spectral decomposition, property (β), decomposable operator, eigen-distribution, sheaf model.

Mathematics Subject Classification (2000) **:** 47B40.

1. If one would make a rigorous selection from the works devoted to linear operator theory, then, for sure, the article [9] of Erret Bishop deserves to stay in line with the classical references of this domain. Together with [8] they represent the only Bishop's contributions (made at the beginning of his mathematical career) to the domain of abstract spectral analysis of linear operators.

The studies of this gifted mathematician dedicated to a variety of topics (from rational approximation theory and uniform algebras, interpolation theory and several complex variables, to the reevaluation of mathematical analysis on constructive grounds) produced deep theorems and original new paths of thought. Bishop was in his heart an analyst endowed with that global understanding and intuition of the fundamentals facts that go further than a simple usage of the technical tools. He died prematurely, after a long period of silence, in 1983.

It is not our aim to present the mathematical work of Erret Bishop. We confine ourselves to offer a succinct analysis of the paper [9] as a moment of synthesis and a starting point for new vistas in the abstract theory of linear operators.

[1]Text of a conference given by the author in 1988, at the Institute of Mathematics of the University of Bucharest.

2. Illustrating a general tendency of the post-war mathematics, the problems of classification and description of the internal structure of abstractly defined classes of linear operators was a popular topics in the two decades after the Second World War. Even though, as seen from today, the motivation for such researches is lost in the early periods of functional analysis or quantum mechanics, some of the problems raised then have intrinsic value and a long lasting life. Such a question makes the subject of the present note.

To begin with we recall some well known fact about the classification and structure of linear operators, or even finite rank matrices.

Let H be a Hilbert space of *finite dimension* and let us consider a linear map $T : H \to H$. The Jordan canonical form offers a complete classification (up to similarity). More precisely, there exists a base of H (not necessary orthonormal) with respect to which T is represented by the following matrix:

$$T \sim \begin{pmatrix} \begin{matrix} \lambda & 1 & \dots & 0 \\ & \lambda & 1 & \dots \\ & & \dots & 1 \\ & & & \lambda \end{matrix} & 0 & 0 & 0 \\ 0 & \begin{matrix} \mu & 1 & & \\ & \mu & 1 & \\ & & \dots & 1 \\ & & & \mu \end{matrix} & 0 & 0 \\ 0 & 0 & \dots. & \\ 0 & 0 & 0 & \begin{matrix} \nu & 1 & & \\ & \nu & 1 & \\ & & \dots & 1 \\ & & & \nu \end{matrix} \end{pmatrix}. \tag{1}$$

The complex numbers $\lambda, \mu, ..., \nu$ (which are not necessary distinct) build up the spectrum of T, $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not invertible}\}$. The base of the space H for which the representation (1) is valid consists on the generalized proper vectors. To be more precisely, each block from the matrix (1) corresponds to the subspace of H generated by the vectors $(\lambda - T)^{p-1}\xi$, ..., $(\lambda - T)\xi$, ξ, where $(\lambda - T)^p\xi = 0$. In this manner, the knowledge of the solutions $\xi \in H$ of the spectral problem

$$(\lambda - T)^p\xi = 0, \quad \lambda \in \mathbb{C},\, p \geq 1, \tag{2}$$

completely determines the operator T.

The second well-known example of spectral decomposition that provides a remarkable classification is connected with the *self-adjoint operators*. Let us consider a separable complex Hilbert space and a bounded linear operator $A \in L(H)$. A is called self-adjoint if $A = A^*$. A classical result, proved independently, at the beginning of the 20th century, by Hahn and Hellinger (see [34]), states that the operator A in unitarily equivalent to a canonical model:

$$A \sim \overset{N}{\underset{k=1}{\oplus}} A_k,\ H \cong \overset{N}{\underset{k=1}{\oplus}} L^2(\mu_k), \tag{3}$$

where $N \in \mathbb{N} \cup \{\infty\}$, A_k represents the multiplication by $x \in \mathbb{R}$ on the space $L^2(\mu_k)$, and μ_k is a positive Borel measure, having compact support on $\mathbb{R}$.

Reducing at a single cell of the representation (3), i.e. at the operator A_k (which is called cyclic), we note that there exists a bijective correspondence between the class of unitary equivalence of A_k and the measure μ_k (up to a natural mutual domination equivalence relation). Informally speaking, the classification of cyclic self-adjoint operators means in fact the classification of the Borel measures.

The attempt to represent self-adjoint operators in a base consisting of (generalized) proper vectors fails for the simple reason that, in this case, equation (2) could not possess enough many solutions. For example, the multiplication by $x \in \mathbb{R}$ on the space $L^2([0,1], dx)$ is an operator having no generalized proper vectors. P. A. M. Dirac has solved this abnormality by using some imaginary proper vectors which exist on the continuous spectrum of the respective self-adjoint operator (see [14]). In the modern terminology Dirac noted that the distributions of the form

$$u(x,y) = f(y)\delta(x-y) \in \mathcal{D}'(\mathbb{R}) \hat{\otimes} L^2(\mathbb{R}, \mu_k)$$

satisfy the equation

$$(\lambda - A_k)u = (\lambda - x)u(\lambda, x) = 0. \tag{4}$$

Moreover, there are enough such proper distributions u so that the vectors $u(1)$ generate the domain of the given operator. If one knows enough solutions in the distribution sense for equation (4), then the operator A_k can be constructed and classified by the matrix associated to these elements (method which is a precursor of the kernel distribution theory). This is the framework in which, for the first time, Dirac's distribution δ has appeared.

A classification similar with the one of self-adjoint operators is the one for *normal operators*, appearing for instance in the work of von Neumann (see [34]). A normal operator N consists of a pair of self-adjoint operators which commute: $N = A + iB$, $[A, B] = 0$. In this case the measures from (3) have compact supports in the complex plane; the same is valid for the proper distributions (4).

The classification of *commutating systems of self-adjoint operators* is similar. The existence of proper common distributions leads to a simultaneous "diagonalization" of these operators and therefore to a classification of them.

3. The idea of diagonalization (and the inherent classification by this method) of non self-adjoint operators by using proper distributions is widespread starting with the coming into being and development of distribution and kernel distribution theory in the period 1950-1960. From this point of view not only some special classes of differential operators (L. Garding, M.V. Kedîş, B.M. Levitan, I.M. Gelfand - A.G. Kostyuchenko), but also some classes of abstract linear operators (F.E. Brower, N. Dunford, I. Gohberg - M.G. Krein, E. Nelson) have been studied. The monographs [7] and [26] treat systematically the subject. Let us note that these attempts continue a valuable classical tradition going back via the works of Titchmarsh [36] and Weyl [38] to XIX-th century.

In order to present just one example, we recall that Brower [11] proves a result that, in the context of the above considerations, can be stated in the following way: let $N \in L(H)$ be a normal operator and $K \subseteq H$ a closed subspace which is invariant for N. Then $S = N|_K$ (which is called *subnormal operator*) can be diagonalized by a system of proper distributions of S^*.

Along the same lines, the Jordan model (1) of operators on finite dimensional spaces has been extended in several ways to operators on normed spaces. Simply mention N. Dunford who has started the axiomatic study of operators more general than the normal ones. These operators are called *spectral operators* and they cover numerous examples imposed by applications (see [16] and [17]). Briefly, a *(Dunford) spectral operator* $T = S + Q$ has a quasinilpotent part Q, a scalar part S which commutes with Q and

$$S = \int_{\mathbb{C}} zE(dz), \tag{5}$$

where E is a measure (taking values on the set of bounded operators) strongly continuous, having compact support in $\mathbb{C}$ and satisfying the following multiplication rule:

$$E(\sigma \cap \delta) = E(\sigma)E(\delta), \quad \sigma, \delta \in Borel(\mathbb{C}).$$

The representation (5) synthesizes the diagonalization of the scalar operator S by the subspaces $E(\delta)H$, $\delta \in Borel(\mathbb{C})$. It is obvious that each normal operator is a scalar operator.

Among other facts formula (5) shows that the continuous analogue of the space consisting of (generalized) proper vectors associated to a proper value from the finite dimensional case is (in Dunford's framework) a subspace having the form $E(\delta)H$ corresponding to a Borel masurable subset δ of the spectrum of T. This far reaching interpretation of proper vectors spaces was essential for the development of the mathematical foundations of quantum mechanics.

Notice two properties of the subspaces $E(\delta)H$ (called *spectral subspaces*):

$$\sigma(S, E(\delta_j)H) \subseteq \overline{\delta_j}, \qquad \sum_j E(\delta_j)H = H, \tag{6}$$

for each finite system (δ_j) of Borel sets which cover the complex plane.

An immediate observation, obtained exactly as in the case of normal operators, shows that each scalar operator S has enough proper distributions. More precisely, $E(\delta)H$ is the subspace generated by the vectors $u(\varphi)$, where $u \in \mathcal{D}'(\mathbb{C}, H)$, $(S - z)u = 0$, $\text{supp}(u) \subseteq \delta$, $\varphi \in \mathcal{D}(\mathbb{C})$, δ being a closed subset of $\mathbb{C}$.

An exhaustive presentation of the theory of scalar and spectral operators can be found in the monograph [17].

4. Chronologically speaking, at this moment Bishop's thesis, which was published in the articles [8] and [9], comes into light. Even though it seems as having no connections with the above mentioned framework, the thesis contains a synthesis (obtained by huge conceptualization steps) of the main phenomenona concerning spectral decompositions. Bishop was able to condense and explain in a couple of theorems all known results, plus a series of facts judged at that time to be pathological. He has also clearly marked the path for all future research in the domain.

Next we briefly analyze the contents of [9].

The author aims at *"seeking a spectral theory which will be valid independently of any of the usual restrictions (such as normality or complete continuity). It is, of course, not to be expected, in view of many known counter examples, that such a theory will even approach in power the spectral theory of a Hermitian or normal operator on a Hilbert space. In fact, it is surprising that a spectral theory for an arbitrary operator exists at all. The results obtained here are incomplete, but it seems likely that any spectral theory which is valid for an arbitrary operator will be closely related to the theory developed here."* ([9], page 379).

Now, more than 50 years after the publication of these prophetic words, we can only admire and confirm their value.

Let T be a bounded linear operator on a Hilbert space H. One wants to find subpaces of H which induce a decomposition of T based on the regions of its spectrum as in (6). Bishop considers two natural choices for these subspaces and we present below only one of them. Let $F \subseteq \mathbb{C}$ be a closed set and $M(F,T)$ the kernel of the natural function $J_F(\xi) = [1 \otimes \xi]$:

$$0 \to M(F,T) \to H \stackrel{J_F}{\to} \mathcal{O}^H(\mathbb{C} \setminus F)/(z-T)\mathcal{O}^H(\mathbb{C} \setminus F). \qquad (7)$$

Here $\mathcal{O}^H(U)$ denotes the Fréchet space of analytic functions on U taking values in the Hilbert space H.

One can easily check that $M(F,T) = E(F)H$, provided that T is a normal or scalar operator with associated spectral measure E.

Bishop distinguishes four grades of spectral decompositions which are called duality theories of type $1-4$. One of the main theorem of the paper states that each operator admits a duality theory of type 4, the other conditions being more restrictive. By definition, an operator T admits a *duality theory of type 4* if the subspace $\sum_j M(\overline{U_j}, T) \subseteq H$ is dense in H, for each finite cover (U_j) of $\mathbb{C}$ consisting of open sets.

It is Bishop's merit to have discovered the connections between spectral decomposition properties of the operator T and the behavior of the linear function $z - T^{'}$, where $T^{'}$ is the $\mathbb{C}$-linear and continuous conjugate of T. Bishop carries forth four kinds of behaviors for $T^{'}$, named as conditions $(\alpha) - (\delta)$. The operator $T^{'}$ is said to satisfy *condition* (β) if the function

$$z - T^{'} : \mathcal{O}^H(U) \to \mathcal{O}^H(U)$$

is injective and has closed range, for each open set $U \subseteq \mathbb{C}$.

Theorem 5 of [9] asserts that an operator T admits a duality theory of type 2 if $T^{'}$ satisfies condition (β).

A proof of this result can be obtained by considering the dual of the exact sequence (7) and using the resolubility of the additive Cousin problem with respect to the cover (U_j). The crucial points of the proof are the space identification

$$[\mathcal{O}^H(U)/(z-T)\mathcal{O}^H(U)]^{'} \cong Ker(z - T : \mathcal{O}^H(U)^{'} \to \mathcal{O}^H(U)^{'})$$

and the remark that the second space contains the proper analytical functionals u having support in a closed subset of U. Such a proper functional inherits the division property:

$$u(\mathbf{1}) = (\lambda - T)u(\frac{1}{\lambda - z}),\ \lambda \notin \overline{U},$$

therfore $(J_{\mathbb{C}-U})^{'}u = u(\mathbf{1}) \in M(\mathbb{C} - U, T)$.

In other words, similarly to the phenomena presented above in the case of proper distributions, one can deduce that the dual of an operator satisfying condition (β) has enough many proper analytic functionals.

In this way one can see that the operators admitting a duality theory of type 2 (or 3) have spectral decompositions of type (6) (with the second condition relaxed).

In the final part of his paper Bishop put into practice his new results in some situations which are difficult to deal by any other means.

5. Bishop did not continue his study of spectral decompositions and had no collaborators to carry forward his ideas. The natural continuation of his research was taken over by C. Foiaş who, among other things, inspired by the duality theory of type 2, has introduced a new class of operators [23].

To be more specific, an operator $T \in L(H)$ is called *decomposable* if it satisfies condition (β) and

$$\sum_j M(\overline{U_j}, T) = H,$$

provided that (U_j) is a finite cover of the complex plane with open sets.

The merit to continue Bishop's ideas rests with Foiaş's school. They framed an abstract and comprehensive theory for spectral decompositions. We mention here the remarkable contributions of C. Apostol, I. Bacalu, I. Colojoară, Şt. Frunză, F. -H. Vasilescu etc. This first stage of development of Foiaş group is amply recorded and comented in the monograph [17].

In the seventies decomposable operators gain maturity and respect, with the twist that at that time almost all abstract theories were abandoned for making room to the study of individual, concrete operators. We mention only two instances, namely the papers of E. Albrecht - that bring a new trend due to his discovery of pathological examples of decomposable operators [1] and his far reaching simplification of their definition [2] - and M. Radjabalipour's paper [31] which contains numerous open problems that will guide the ulterior research work. A synthesis of the results concerning decomposable operators can be found in F. -H. Vasilescu's book [37].

Coming back to Bishop's original idea, that is to study spectral decompositions in conjunction with duality, Şt. Frunză and M. Radjabalipour made important progress. In the same time Frunză has initiated the axiomatic study of spectral decompositions for systems of commuting operators [25].

Even though this presentation could be (and deserve to be) continued, we shall stop here, not before inviting the reader that is eager for more details to consult the reference list.

6. Some recent results (in 1988 !) fill the present picture of operators satisfying Bishop's condition (β). In particular, the problems stated at the beginning of the present note, namely the classification and diagonalization of operators by proper distributions, start to get a coherent answer for operators satisfying condition (β). A variety of classes of operators that are currently studied satisfy condition (β).

First it has been understood the significance of the condition (β) in connection with the existence of extensions of the original operator to a "better" one (a recurrent theme in operator theory). More precisely, it is known that an operator satisfies condition (β) (respectively (β) on smooth functions) if and only if it is the restriction of a decomposable operator (respectively one possessing a functional calculus with smooth functions) on a closed invariant subspace (see [4] and [22]).

Second, we have proved that an operator T is decomposable if and only if T and T' satisfy condition (β) (see [27], [21]).

According to an amazing observation originally stated for subnormal operators (due to S. Brown) it was proved that an operator satisfying condition (β) and having interior points into the spectrum has nontrivial invariant subspaces (see [10] and [20]). Moreover, if an operator satisfying condition (β) has interior point into the essential spectrum, then it has a lattice of invariant subspace which is richer that the lattice of all closed subspaces of an separable Hilbert space (see [3]).

All these facts disclose, as a confirmation of Bishop's ideas, the richness of the spectral theory of operators satisfying condition (β). The next step along the lines presented above would be a classification of operators satisfying condition (β) by their spectral data. A first step in this direction is the *sheaf model* of an operator satisfying this very condition (this model is presented in [29]). More precisely, the map

$$U \to \mathcal{F}(U) = \mathcal{O}^H(U)/(z-T)\mathcal{O}^H(U)$$

is an analytic sheaf of Fréchet spaces, supported by the spectrum of the operator T. The multiplication by the variable z on the space of global sections $\mathcal{F}(C) \cong H$ can be identified with the action of the operator T on H. This sheaf (which implicitly appears in Bishop's thesis - see reference [7]) synthesizes the spectral information about T. To give a simple example, the operator T is decomposable if and only if the sheaf F is soft (see [29]).

For more restricted classes of operators satisfying condition (β) there exist nowadays functional models and more elaborated classifications.

7. Finally, in order to resume the link to distribution theory, we mention that abstract reformulations of the condition (β) lead to nontrivial results concerning the division of vector valued distributions by analytic functions (see [22]).

For instance, let us consider a space Ω endowed with a positive measure and let us fix $p \in [1, \infty]$. Then, the polynomial map

$$P(z,\omega) : \mathcal{D}'(\mathbb{C}) \hat{\otimes} L^p(\Omega) \to \mathcal{D}'(\mathbb{C}) \hat{\otimes} L^p(\Omega),$$

where

$$P(z,\omega) = z^n + a_1(\omega)z^{n-1} + ... + a_n(\omega), \quad a_j \in L^\infty(\Omega), \, 1 \le j \le n,$$

is onto.

Similarly, one can prove that the application

$$P(t,\omega) : \mathcal{D}'(\mathbb{R}) \hat{\otimes} L^p(\Omega) \to \mathcal{D}'(\mathbb{R}) \hat{\otimes} L^p(\Omega)$$

is onto if and only if

$$\operatorname{ess} - \sup_{\omega \in \Omega}(\max\{|\operatorname{Im}\lambda|^{-1} : P(\lambda,\omega) = 0, \, \lambda \notin \mathbb{R}\}) < \infty.$$

Some applications of these division lemmas are presented in the paper [22] and one of its continuation.

Acknowledgements

The author is grateful to Professor Radu Miculescu for rescuing the text of this conference and offering to translate and publish it in the present volume.

References

[1] E. Albrecht, On two questions of I. Colojoară and C. Foiaş, *Manuscripta Math.,* **25** (1978), 1-15.

[2] E. Albrecht, On decomposable operators, *Integral Equations Oper. Theory*, **2** (1979), 1-10.

[3] E. Albrecht and B. Chevreau, Invariant subspaces for l^1-operators having Bishop's property (β) on a large part of their spectrum, *J. Oper. Theory*, **18** (1987), 339–372.

[4] E. Albrecht and J. Eschmeier, Functional models and local spectral theory, *Proc. Lond. Math. Soc.*, III. Ser., **75** (1997), 323-348.

[5] C. Apostol, Spectral decompositions and functional calculus, *Rev. Roum. Math. Pures Appl.,* **13** (1968), 1481-1528.

[6] C. Apostol, The spectral flavour of Scott Brown's techniques, *J. Oper. Theory*, **6** (1981), 3-12.

[7] Iu.M. Berezanskii, *Descompunerea operatorilor autoadjuncţi după funcţii proprii* (în lb. rusă), Kiev, 1965.

[8] E. Bishop, Spectral theory for operators on a Banach space, *Trans. Am. Math. Soc.*, **86** (1957), 414-445.

[9] E. Bishop, A duality theorem for an arbitrary operator, *Pac. J. Math.*, **9** (1959), 379-397.

[10] S. Brown, Hyponormal operators with thick spectra have invariant subspaces, *Ann. Math.*, **125** (1987), 93-103.

[11] F. E Browder, Eigenfunction expansions for non-symmetric partial differential operators, II, *Am. J. Math.,* **81** (1959), 1-22.

[12] I. Colojoară and C. Foiaş, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.

[13] M. Cowen and R.G. Douglas, Complex geometry and operator theory, *Acta Math.*, **141** (1978), 187-261.

[14] P.A.M. Dirac, *The principles of quantum mecahnics*, Clarendon Press, Oxford, IV-th edition, 1958.

[15] N. Dunford, Spectral operators, *Pac. J. Math.*, **4** (1954), 321-354.

[16] N. Dunford, A survey of the theory of spectral operators, *Bull. Am. Math. Soc.*, **64** (1958), 217-274.

[17] N. Dunford and J. Schwartz, *Linear operators*, Part III, Wiley-Interscience, New York, 1971.

[18] I. Erdelyi and R. Lange, *Spectral decompositions on Banach spaces*, Lectures Notes in Math., vol. 623, Springer, Berlin-Heidelberg-New York, 1977.

[19] J. Eschmeier, *Analytische Dualität und Tensorprodukte in der mehr dimensionalen Spektraltheorie*, Habilitationsschrift, Münster, 1986.

[20] J. ESCHMEIER, Bishop's condition (β) and joint invariant subspaces, *J. Reine Angew. Math.*, **426** (1992), 1-22.

[21] J. ESCHMEIER and M. PUTINAR, Spectral theory and sheaf theory. III, *J. Reine Angew. Math.*, **354** (1984), 150-163.

[22] J. ESCHMEIER and M. PUTINAR, Bishop's condition (β) and rich extensions of linear operators, *Indiana Univ. Math. J.*, **37** (1988), 325-348.

[23] C. FOIAŞ, Spectral maximal spaces and decomposable operators in Banach spaces, *Arch. Math.*, **14** (1963), 341-349.

[24] C. FOIAŞ, On the maximal spactral spaces of a decomposable operator, *Rev. Roum. Math. Pures Appl.,* **15** (1970), 1599-1606.

[25] ŞT. FRUNZĂ, O teorie axiomatică a descompunerilor spectrale pentru sisteme de operatori. I, *Stud. Cercet. Mat.*, **27** (1975), 655-711; II, ibidem **29** (1977), 329-376.

[26] I.C. GOHBERG and M.G. KREIN, *Introduction to the theory of linear nonselfadjoint operators*, Translations of Mathematical Monographs. 18. Providence, RI: American Mathematical Society (AMS). XV, 378 p., 1969.

[27] R. LANGE, A purely analytic criterion for a decomposable operator, *Glasg. Math. J.*, **21** (1980), 69-70.

[28] J.D. PINCUS, Commutators and systems of singular integral equations. I, *Acta Math.*, **121** (1968), 219-249.

[29] M. PUTINAR, *Spectral theory and sheaf theory. I, Dilation theory, Toeplitz operators, and other topics*, 7th int. Conf. Oper. theory, Timisoara and Herculane/Rom. 1982, Operator Theory: Advances and Applications, vol. 11, pp. 283-297, 1983.

[30] M. PUTINAR, Extensions scalaires et noyaux distribution des opérateurs hyponormaux, *C. R. Acad. Sci., Paris*, Sér. I, **301** (1985), 739-741.

[31] M. RADJABALIPOUR, Decomposable operators, *Bull. Iranian Math. Soc.*, **9** (1978), 1-49.

[32] M. REED and B. SIMON, *Methods of modern mathematical physics. Part I*, Academic Press, New York, 1972.

[33] J. SNADER, Bishop's condition (β), *Glasg. Math. J.*, **26** (1985), 35-46.

[34] M.H. STONE, *Linear transformations in Hilbert spaces and their applications to annalysis*, Amer. Math. Soc. Colloq. Publ. 15, Providence, 1932.

[35] B.SZ. NAGY and C. FOIAŞ, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Akad. Kiado, Budapest, 1967.

[36] E.C. TITCHMARCH, *Eigenfunction expansions associated with second-order differential equations*, Claderon Press, Oxford, 1946.

[37] F.-H. VASILESCU, *Analytic functional calculus and spectral decompositions*, D. Reidel Co., Dordrecht, 1982.

[38] H. WEYL, Ramifications old and new, of the eigenvalue problem, *Bull. Am. Math. Soc.*, **56** (1950), 115-139.

[39] D. XIA, Analytic theory of subnormal operators, *Integral Equations Oper. Theory*, **10** (1987), 880-903.

Mihai Putinar
University of California at Santa Barbara
Santa Barbara, CA 93106-3080, USA
E-mail: `mputinar@math.ucsb.edu`