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Polynomial approximation and generalized Toeplitz operators

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Communicated by Petru Mironescu

This paper is dedicated to Professor Ion Colojoară on the occasion of his 80th birthday

Abstract - Let μ be a compactly supported finite positive Borel measure in the plane and for any $t \in [1, \infty)$ let $P^t(\mu)$ be the closure in $L^t(\mu)$ of all analytic polynomials. We show that the set $\{|p|^t : p \in P^t(\mu)\}$ is norm dense in the positive cone of $L^1(\mu)$ if and only if μ vanishes on the set $abpe(P^t(\mu))$ of all analytic bounded point evaluations of $P^t(\mu)$. In the case when $P^t(\mu)$ is irreducible, μ is shown to have this property if and only if it is of the form $d\mu = (|g|^t dm) \circ \psi^{*-1}$ where m is the normalized Lebesgue measure on the unit circle, ψ^* is the boundary value of some univalent function $\psi \in H^\infty$ and $g \in H^t$ is a cyclic vector for the analytic Toeplitz operator T_{ψ^*} on the Hardy space H^t. As an application we show that a cyclic irreducible subnormal operator S satisfies a Hartman-Wintner type spectral inclusion condition if and only if it is unitarily equivalent to the multiplication by z on the Hardy space $H^2(G)$ where $G = \sigma(S) \backslash \sigma_e(S)$.

Key words and phrases : subnormal operators, bounded point evaluation, harmonic measure, Hardy space, spectral inclusion.

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1. Introduction

Let μ be a positive finite Borel measure in the complex plane with compact support. Let $t \in [1, \infty)$ and let $P^t(\mu)$ denote the closure in $L^t(\mu)$ of all analytic polynomials. A point $\lambda \in \mathbb{C}$ is said to be a bounded point evaluation for $P^t(\mu)$ if the mapping $p \mapsto p(\lambda)$ defined on polynomials extends to a continuous linear functional on $P^t(\mu)$. In this case, for each $f \in P^t(\mu)$ one denotes by $\hat{f}(\lambda)$ the value of this extension applied to f. The set of all bounded point evaluations for $P^t(\mu)$ is denoted by $bpe(P^t(\mu))$. A point $\lambda \in \mathbb{C}$ is called an analytic bounded point evaluation for $P^t(\mu)$ if there exists an open disc U around λ such that $U \subset bpe(P^t(\mu))$ and moreover such that for each $f \in P^t(\mu)$ the mapping $\zeta \mapsto \hat{f}(\zeta)$ is analytic on U. The set of all analytic bounded point evaluations for $P^t(\mu)$ is denoted by $abpe(P^t(\mu))$. Basic facts about bounded point evaluations can be found in Sec. II.7 in [9].

In this paper we are interested in the following approximation problem. Find conditions under which every nonnegative function $f \in L^1(\mu)$ can be approximated in the $L^1(\mu)$ norm by a sequence of the form $\{|p_n|^t\}_{n\geq 1}$ where each p_n is an analytic polynomial. We shall provide a description of those measures having this approximation property. In particular, it will follow that a measure μ has this property if and only if it vanishes on its set of analytic bounded point evaluations. Our proofs are essentially based on the fundamental results of J. Thomson (see [17]) describing the analytic structure of the spaces $P^t(\mu)$. We also make use of the reduction to the unit disc devised by R. Olin and L. Yang (see [13]).

As an application, we shall provide a characterization of those cyclic irreducible subnormal operators for which their associated Toeplitz operators enjoy a spectral inclusion property similar to the one appearing in the classical case that has been proved by P. Hartman and A. Wintner in [10]. We show that every such operator S is unitarily equivalent to the multiplication by z on the Hardy space $H^2(G)$ where $G = \sigma(S)\setminus\sigma_e(S)$.

2. Preliminaries

Let us consider a positive finite Borel measure μ with compact support in the plane. The space $P^t(\mu)$ is called pure if it does not have nontrivial L^t summand. The space $P^t(\mu)$ is said to be irreducible if it does not contain nontrivial characteristic functions. If $G \subset \mathbb{C}$ is a bounded open set then $H^\infty(G)$ denotes the Banach algebra of all bounded analytic functions on G. The following theorem summarizes the basic results from [17] that will be needed in the sequel:

Theorem 2.1. (cf. [17]) *Let μ be a compactly supported positive finite Borel measure in the complex plane and let supp(μ) denote its closed support. Let $t \in [1,\infty)$. Then there exists a Borel partition $\{\Delta_n\}_{n\geq 0}$ of supp(μ) such that the space $P^t(\mu)$ admits the direct sum decomposition*

$$P^t(\mu) = L^t(\mu_0) \oplus (\bigoplus_{n\geq 1} P^t(\mu_n))$$

(where μ_n is the restriction of μ to Δ_n) such that for each $n \geq 1$ the space $P^t(\mu_n)$ has the following properties:

(1) $P^t(\mu_n)$ is irreducible;

(2) if $W_n = abpe(P^t(\mu_n))$, then W_n is a simply connected region and its closure contains Δ_n;

(3) the mapping $f \mapsto \hat{f}$ is one-to-one on $P^t(\mu_n)$. Moreover, $f = \hat{f}$ μ_n-a.e. on W_n for every $f \in P^t(\mu_n)$;

(4) *the mapping $f \mapsto \hat{f}$ implements an isometric isomorphism and a weak* homeomorphism between the dual Banach algebras $P^t(\mu_n) \cap L^\infty(\mu_n)$ and $H^\infty(W_n)$.*

We shall also need several results from [13] that are summarized ln Theorem 2.2 below. We first need some notations. Let $t \in [1, \infty)$. Let N_μ denote the multiplication by z on $L^t(\mu)$ and let S_μ denote its restriction to $P^t(\mu)$. If μ is a measure such that $P^t(\mu)$ is irreducible and $G = abpe(P^t(\mu))$ then for each $f \in H^\infty(G)$ one denotes by $\tilde{f} \in P^t(\mu) \cap L^\infty(\mu)$ its image under the isomorphism between $H^\infty(G)$ and $P^t(\mu) \cap L^\infty(\mu)$ appearing in Theorem 2.1. In the case of the normalized Lebesgue measure m on the unit circle, we shall employ the usual notation H^t (instead of $P^t(m)$) for the classical Hardy space on $\partial\mathbb{D}$. Only in this case, the same letter will be used both for the space of boundary values and for that of their analytic extensions, we hope without confusion. For every $\psi \in H^\infty(\mathbb{D})$ we shall denote by ψ^* its boundary value that is

$$\psi^*(e^{i\theta}) = \lim_{r\to 1} \psi(re^{i\theta})$$

which exists m-a.e. on $\partial\mathbb{D}$.

Let $\{\Omega, \mathcal{B}, \mu\}$ be a measure space and let $\mathcal{B}'$ be a σ-algebra of subsets of some set Ω'. If $f : \Omega \to \Omega'$ is a measurable mapping then one denotes by $\mu' = \mu \circ f^{-1}$ the measure on Ω' defined by $\mu'(\sigma) = \mu(f^{-1}(\sigma))$ for every $\sigma \in \mathcal{B}'$ (the push-forward measure).

Theorem 2.2. (cf. [13]) *Let $t \in [1, \infty)$ and let μ be a compactly supported positive finite Borel measure in the plane such that $P^t(\mu)$ is irreducible. Let $G = abpe(P^t(\mu))$ and let $\varphi : G \to \mathbb{D}$ be a conformal mapping onto the unit disc and let $\psi = \varphi^{-1} : \mathbb{D} \to G$. Let ν be the measure on the closed unit disc defined by $\nu = \mu \circ \tilde{\varphi}^{-1}$. Then the following hold:*

(1) *$|\tilde{\varphi}(\zeta)| = 1$ μ-a.e. on ∂G;*

(2) *the space $P^t(\nu)$ is irreducible and $abpe(P^t(\nu) = \mathbb{D}$;*

(3) *$\mu = \nu \circ \tilde{\psi}^{-1}$ (where $\tilde{\psi} \in P^t(\nu) \cap L^\infty(\nu)$ is the image of ψ under the isomorphism appearing in Theorem 2.1);*

(4) *the mapping $p \mapsto p \circ \tilde{\psi}$ extends to a unitary operator (surjective isometry) $U : P^t(\mu) \to P^t(\nu)$;*

(5) *if $T_{\tilde{\psi}}$ is the multiplication by $\tilde{\psi}$ on $P^t(\nu)$ then $T_{\tilde{\psi}} U = U S_\mu$;*

(6) *ν restricted to $\partial\mathbb{D}$ is absolutely continuous with respect to the Lebesgue measure and $\tilde{\psi} = \psi^*$ ν-a.e. on $\partial\mathbb{D}$;*

(7) the restriction to $\partial\mathbb{D}$ of the mapping $\tilde{\psi}$ is ν-a.e. one-to-one from a carrier of $\nu|_{\partial\mathbb{D}}$ onto a carrier of $\mu|_{\partial G}$.

We shall now recall some elementary definitions related to Hardy spaces and harmonic measures on bounded simply connected planar domains. Let G be such a domain, let $t \in [1, \infty)$, and let $\alpha \in G$. The Hardy space $H^t(G)$ is the set of all analytic functions f on G for which $|f|^t$ has a harmonic majorant on G. If $f \in H^t(G)$ and u_f is the least harmonic majorant of $|f|^t$, then one denotes $\|f\|_\alpha = u_f(\alpha)^{1/t}$. It turns out that $\{H^t(G), \|\cdot\|_\alpha\}$ is a Banach space. If $\beta \in G$ is another point, then the two corresponding norms are equivalent. If $\psi : \mathbb{D} \to G$ is any conformal mapping with $\psi(0) = \alpha$, then the mapping $W(f) = f \circ \psi$ implements an isometry from $H^t(G)$ onto $H^t(\mathbb{D})$ such that $T_\psi W = WM_z$, where T_ψ is the multiplication by ψ on $H^t(\mathbb{D})$ and M_z is the multiplication by z on $H^t(G)$.

Suppose again that G is a bounded simply connected planar domain. For each $z \in G$ let $\psi_z : \mathbb{D} \to G$ be a conformal mapping such that $\psi_z(0) = z$ and let $\omega_z = m \circ \psi_z^{*-1}$. Then ω_z is a probability measure on ∂G and it is called the harmonic measure of G at z. (This is only one of the several equivalent definitions of the harmonic measure.) If $z, w \in G$ then ω_z and ω_w are mutually boundedly absolutely continuous, by Harnack's inequality. Suppose now that ω_α is the harmonic measure of G at a fixed point $\alpha \in G$. Then for each $f \in L^1(\omega_\alpha)$ the function

$$\mathcal{P}[f](z) = \int_{\partial G} f d\omega_z, \quad z \in G$$

is harmonic on G and it gives the Perron-Wiener-Brelot solution of the (generalized) Dirichlet problem for G. If f is continuous, then $\mathcal{P}[f]$ extends continuously to ∂G and its boundary value agrees with f everywhere on ∂G. Moreover, for each $t \in [1, \infty)$ we have that $G \subset abpe(P^t(\omega_\alpha))$ and the mapping $f \mapsto \mathcal{P}[f]$ is an isometry from $P^t(\omega_\alpha)$ onto a closed subspace of the Hardy space $H^t(G)$ when the latter is endowed with the $\|\cdot\|_\alpha$ norm. In particular this implies that the space $P^t(\omega_\alpha)$ is irreducible. For a quick introduction on Hardy spaces on planar domains and harmonic measures see Sections V.9 and V.10 in [9].

3. The main result

The main result of this paper is the following:

Theorem 3.1. *Let $t \in [1, \infty)$ and let μ be a compactly supported Borel positive finite measure in the plane. Let m be the normalized Lebesgue measure on $\partial\mathbb{D}$. The following are equivalent:*

(1) $P^t(\mu)$ is irreducible and $\mu(abpe(P^t(\mu))) = 0$;

(2) There exist a bounded simply connected planar domain G, a conformal mapping $\psi : \mathbb{D} \to G$ and a function $g \in H^t$ such that g is a cyclic vector for the analytic Toeplitz operator T_{ψ^*} on H^t and such that $\mu = \nu \circ \psi^{*-1}$ where $d\nu = |g|^t dm$ and $\psi^* \in L^\infty(m)$ stands for the nontangential boundary value of ψ.

Moreover, if (2) holds true, then $G = abpe(P^t(\mu))$.

Proof. In this proof we shall freely use the notations and the results from Theorems 2.1 and 2.2. The word unitary operator will mean a linear surjective isometry.

(1) $\Rightarrow$ (2)

Suppose that $P^t(\mu)$ is irreducible and that $\mu(G) = 0$ where $G = abpe(P^t(\mu))$. Let $\psi : \mathbb{D} \to G$ be a conformal mapping and let $\psi^* \in L^\infty(m)$ be its nontangential boundary value. Let $\varphi = \psi^{-1} : G \to \mathbb{D}$ and let $\nu = \mu \circ \tilde{\varphi}^{-1}$. Since $|\tilde{\varphi}| = 1$ μ-a.e. on ∂G and $\mu(G) = 0$ it follows that $\nu(\mathbb{D}) = 0$ and therefore $supp(\nu) \subset \partial\mathbb{D}$. From Theorem 2.2 we know that $P^t(\nu)$ is irreducible and that $abpe(P^t(\nu) = \mathbb{D}$. Since $P^t(\nu)$ is irreducible, ν is absolutely continuous with respect to the Lebesgue measure on $\partial\mathbb{D}$. Moreover, since $abpe(P^t(\nu)) = \mathbb{D}$ it then follows from Szego' Theorem that ν has the form $d\nu = hdm$ where $h \in L^1(m)_+$ with $\log h \in L^1(m)$. It then follows that there exists an outer function $g \in H^t$ such that $h = |g|^t$ m-a.e. on $\partial\mathbb{D}$.

Let us show that g is cyclic for T_{ψ^*} on H^t. For this purpose, recall first that by Theorem 2.2 the mapping $p \mapsto p \circ \tilde{\psi}$ extends to a unitary operator $U : P^t(\mu) \to P^t(\nu)$ such that $T_{\tilde{\psi}} U = U S_\mu$. Let $\Gamma : P^t(\nu) \to H^t$ be defined by $\Gamma(f) = fg$ for for every $f \in P^t(\nu)$. Then Γ is isometric and surjective (because g is outer) and $S_m \Gamma = \Gamma S_\nu$ where S_m and S_ν are the corresponding shifts operators. In particular, since $\tilde{\psi} = \psi^*$ ν-a.e., we have $\Gamma(p \circ \tilde{\psi}) = (p \circ \psi^*)g$ for every analytic polynomial p. Finally the mapping $V : P^t(\mu) \to H^t$ defined as $V = \Gamma \circ U$ is therefore a unitary operator and $Vp = (p \circ \psi^*)g$ for every polynomial p, which shows that g is a cyclic vector for T_{ψ^*}. Moreover, since $\tilde{\psi} = \psi^*$ ν-a.e. on $\partial\mathbb{D}$ it follows that μ has indeed the form specified at (2).

(2) $\Rightarrow$ (1)

Suppose now that there exist a bounded simply connected domain G, a conformal mapping $\psi : \mathbb{D} \to G$ and $g \in H^t$ such that g is a cyclic vector for T_{ψ^*} on H^t and such that $\mu = \nu \circ \psi^{*-1}$ where $d\nu = |g|^t dm$. Let $G_1 = abpe(P^t(\mu))$. Since ψ is a conformal mapping we have that $\mathcal{R}(\psi^*) \subset \partial G$ therefore $\mu(G) = 0$ ($\mathcal{R}(f)$ stands for the essential range of f). It remains to show that $G = G_1$.

First, we see that the mapping $p \mapsto (p \circ \psi^*)g$ can be extended to a unitary operator $V : P^t(\mu) \to H^t$ such that $T_{\psi^*} V = V S_\mu$ a fact that follows from the definition of μ and the fact that g is cyclic for T_{ψ^*}. Since T_{ψ^*} is unitarily equivalent to S_μ the map $\Theta(X) = V^{-1} X V$ implements a Banach algebras isomorphism and a weak* homeomorphism between the commutants of T_{ψ^*} and S_μ respectively. Now, since ψ is univalent on $\mathbb{D}$, and since the dual space $(H^t)^*$ is spanned by the eigenvectors of S_m^*, it follows that the commutant of T_{ψ^*} coincides with the algebra of all analytic Toeplitz operators on H^t. On the other hand, Theorem 2.1 implies that the commutant of S_μ on $P^t(\mu)$, which coincides with $P^t(\mu) \cap L^\infty(\mu)$, is isometrically isomorphic with the algebra $H^\infty(G_1)$. Then we see that the mapping Θ defined above induces a dual Banach algebras isomorphism $\tilde{\Theta} : H^\infty(G) \to H^\infty(G_1)$ such that $\tilde{\Theta}(\chi_G) = \chi_{G_1}$ where $\chi_G(z) = z$ on G and similarily for G_1. This easily implies that $G = G_1$. This finishes the proof of this implication and of the theorem as well. $\square$

Corollary 3.1. *Let $t \in [1, \infty)$ and let μ be a compactly supported finite positive Borel measure in the plane. Let $G = abpe(P^t(\mu))$. Then the following are equivalent:*

(1) $\mu(G) = 0$;

(2) for every $f \in L^t(\mu)_+$ such that $f(\zeta) \geq \delta$ μ-a.e. for some $\delta > 0$, there exists $g \in P^t(\mu)$ such that $f = |g|$ μ-a.e.;

(3) for every $f \in L^t(\mu)_+$ there exists a sequence of analytic polynomials $\{p_n\}$ such that $\|f - |p_n|\|_t \to 0$;

(4) for every $f \in L^1(\mu)_+$ there exists a sequence of analytic polynomials $\{p_n\}$ such that $\|f - |p_n|^t\|_1 \to 0$;

(5) the convex hull of the set $\{|p|^t \,|\, p$ analytic polynomial$\}$ is norm dense in the positive cone of $L^1(\mu)$;

(6) for every $\varepsilon > 0$ and every Borel subset $\sigma \subset \mathbb{C}$ with $\mu(\sigma) > 0$ there exists an analytic polynomial p such that

$$\int_{\mathbb{C}\setminus\sigma} |p|^t d\mu < \varepsilon \int_\sigma |p|^t d\mu.$$

Proof. We may assume, by virtue of Theorem 2.1, that $P^t(\mu)$ is irreducible.

(1) $\Rightarrow$ (2) Suppose that $\mu(G) = 0$. It then follows from Theorem 3.1 that $d\mu = d\nu \circ \psi^{*-1}$ where $\psi : \mathbb{D} \to G$ is a conformal mapping and $d\nu = |g|^t dm$ where $g \in H^t$ is a cyclic vector for T_{ψ^*} on H^t. Let $f \in L^t(\mu)_+$ such that $f(\zeta) \geq \delta$ μ-a.e. for some $\delta > 0$ and let $y = (f \circ \psi^*)|g|$. Then $y \in L^t(m)$ and $\log y \in L^1(m)$ therefore there exists $w \in H^t$ such that $y = |w|$ m-a.e. Let $U : P^t(\mu) \to P^t(\nu)$ and $\Gamma : P^t(\nu) \to H^t$ be the unitary operators appearing in the proof of Theorem 3.1. Let $x \in P^t(\mu)$ such that $(\Gamma \circ U)(x) = w$. It is then easy to see that $|x| = f$ μ-a.e. on ∂G.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4) follows from the fact that the mapping $x \mapsto |x|^t$ is continuous from $L^t(\mu)$ into $L^1(\mu)$ when both spaces are endowed with their corresponding norm topologies.

(4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (6) is obvious.

(6) $\Rightarrow$ (1)

It follows from [14, Lemma 2.3] that for any compact set $K \subset abpe(P^t(\mu))$ there exists $c_K > 0$ such that

$$\int_K |f|^t d\mu \leq c_K \int_{supp(\mu)\setminus K} |f|^t d\mu$$

for every $f \in P^t(\mu)$. This easily proves this implication as well.

□

Let us point out that, when $t = 2$, the equivalence between assertions (4) and (6) holds true in a much more general situation in the context of the factorization technique for integrable functions. This technique, initiated in [7] in the context of Hilbert space contractions, has been subsequently refined and extended in an abstract setting in [5] and it played a basic role in the development of the theory of dual algebras.

It was proved in [17] that for any $t \in [1, \infty)$ and for every bounded simply connected domain G there exists a measure μ supported on its closure such that $G = abpe(P^t(\mu))$.

Corollary 3.2. *Let $t \in [1, \infty)$ and let G be a bounded simply connected planar domain. The following are equivalent:*

(1) there exists a measure μ supported on ∂G with the property that $G = abpe(P^t(\mu))$;

(2) if $\psi : \mathbb{D} \to G$ is any conformal mapping, then the analytic Toeplitz operator T_{ψ^*} on H^t has a cyclic vector.

Proof. (1) $\Rightarrow$ (2) Let $\psi : \mathbb{D} \to G$ be any conformal mapping. Suppose there exists a measure μ on ∂G such that $G = abpe(P^t(\mu))$. We may and shall assume that $P^t(\mu)$ is irreducible. It then follows from Theorem 3.1 that the analytic Toeplitz operator T_{ψ^*} has a cyclic vector.

(2) $\Rightarrow$ (1) Conversely, assume that T_{ψ^*} has a cyclic vector $g \in H^t$. Let $d\mu = (|g|^t dm) \circ \psi^{*-1}$. It then follows from Theorem 3.1 and its proof that μ is supported on ∂G and that $G = abpe(P^t(\mu))$.

□

The previous results can be used to provide a characterization of the simply connected domains G for which the analytic polynomials are dense in the Hardy space $H^t(G)$.

Corollary 3.3. *Let G be a bounded simply connected domain in the complex plane and let ω_α be the harmonic measure of G relative to some point $\alpha \in G$. Let $t \in [1, \infty)$. The following are equivalent:*

(1) the space $P^t(\omega_\alpha)$ satisfies any of the six equivalent conditions from Corollary 3.1;

(2) the analytic polynomials are dense in the Hardy space $H^t(G)$.

Proof. It was shown in [15] that condition (2) above is equivalent to the fact that $G = abpe(P^t(\omega_\alpha))$. Let $G_1 = abpe(P^t(\omega_\alpha))$. To show that (1) $\Rightarrow$ (2) we only need to observe that, if $\omega_\alpha(G_1) = 0$, and G is a proper subset of G_1 then it would follow that ω_α vanishes on a relatively open subset of ∂G. This contradicts the fact that the support of the harmonic measure is the whole boundary of G. The converse is obvious, again using the result from [15] quoted above.

□

Simply connected domains G for which M_z is cyclic on $H^t(G)$ have been extensively studied in the literature, see for instance [1], [2], [3], [4], [8] and the references therein.

4. Applications to subnormal operators

In this section we shall give an application of the results obtained above to the study of subnormal operators on Hilbert spaces. We refer to the monograph [9] for a comprehensive exposition of the theory of subnormal operators. Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For each $T \in \mathcal{B}(\mathcal{H})$ one denotes by $\sigma(T)$ its spectrum and by $\sigma_e(T)$ its essential (Calkin) spectrum. Consider a subnormal operator $S \in \mathcal{B}(\mathcal{H})$ and let $N \in \mathcal{B}(\mathcal{K})$ be its minimal normal extension. Let μ be a scalar valued spectral measure for N that is, a probability measure in the plane which is equivalent to the spectral measure of N. We therefore have a von Neumann algebras isomorphism $\pi : L^\infty(\mu) \to W^*(N)$ such that $\pi(z) = N$, where $W^*(N)$ holds for the von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$ generated by N.

Let $P_{\mathcal{H}}$ be the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Then one may define a mapping $\Phi : L^\infty(\mu) \to \mathcal{B}(\mathcal{H})$ by $\Phi(\varphi)h = P_{\mathcal{H}}(\pi(\varphi)h)$ for $\varphi \in L^\infty(\mu)$ and $h \in \mathcal{H}$. One usually denotes $T_\varphi = \Phi(\varphi)$ and T_φ is called the (generalized) Toeplitz operator with symbol φ. This terminology is inspired by the case when S is the unilateral shift on the Hardy space H^2. In this case, a well-known theorem of P. Hartman and A. Wintner [10] asserts that $\sigma(T_\varphi)$ contains the essential range of φ for every $\varphi \in L^\infty(m)$. Let us say that a subnormal operator $S \in \mathcal{B}(\mathcal{H})$ has the Toeplitz spectral inclusion property if $\sigma(T_\varphi)$ contains the essential range of φ for every $\varphi \in L^\infty(\mu)$. For instance, every analytic Toeplitz operator on H^2 has this property. This is because the minimal normal extension of every such operator with nonconstant symbol is the corresponding multiplication operator on $L^2(m)$. This property has been previously studied in [11] and [12]. It follows from the results in [16] that if S is a subnormal operator with the Toeplitz spectral inclusion property, then there exists a *-homomorphism

$$\rho : C^*\{T_\varphi : \varphi \in L^\infty(\mu)\} \to L^\infty(\mu)$$

such that $\rho(T_\varphi) = \varphi$ for every $\varphi \in L^\infty(\mu)$ and whose kernel is the closed ideal generated by all the semi-commutators $T_{\varphi\psi} - T_\varphi T_\psi$ with $\varphi, \psi \in L^\infty(\mu)$. Moreover, it was proved in [11] that S has the spectral inclusion property if and only if, for every Borel subset $\sigma \subset \mathbb{C}$ with $\mu(\sigma) > 0$, we have $\|T_{\chi_\sigma}\| = 1$ where χ_σ holds for the characteristic function of σ. When $S = S_\mu$ on $P^2(\mu)$, this is obviously equivalent to condition (6) in Corollary 3.1.

Before going further, we recall a result of J. Bram [6] asserting that for every cyclic subnormal operator $S \in \mathcal{B}(\mathcal{H})$ there exists a probability Borel measure μ with compact support in the plane such that S is unitarily equivalent to S_μ on $P^2(\mu)$. If S is also irreducible and $G = abpe(P^2(\mu))$, then G is a simply connected domain, $\sigma(S) = \bar{G}$ and $\sigma_e(S) = \partial G$ (see [17]). The results in the previous section allow us to prove the following:

Corollary 4.1. *Let $S \in \mathcal{B}(\mathcal{H})$ be a cyclic irreducible subnormal operator. Then S has the Toeplitz spectral inclusion property if and only if it is unitarily equivalent to M_z on $H^2(G)$ where $G = \sigma(S)\backslash\sigma_e(S)$.*

Proof. Since S is cyclic and irreducible, it can be represented as S_μ on some irreducible space $P^2(\mu)$. Let $G = abpe(P^2(\mu))$.

Suppose that S_μ has the Toeplitz spectral inclusion property. As pointed above, μ satisfies condition (6) in Corollary 3.1. therefore $\mu(G) = 0$. It now follows from Theorem 3.1 and its proof that S_μ is unitarily equivalent to the analytic Toeplitz operator T_{ψ^*} on H^2 where $\psi : \mathbb{D} \to G$ is some conformal mapping. This Toeplitz operator is, at its turn, unitarily equivalent to M_z on the Hardy space $H^2(G)$ when the latter is endowed with the norm $\|\cdot\|_\alpha$. where $\alpha = \psi(0)$.

The converse follows from the fact that every analytic Toeplitz operator on H^2 has the Toeplitz spectral inclusion property. □

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References

[1] J. Akeroyd, Polynomial approximation in the mean with respect to harmonic measure on crescents, *Trans. Amer. Math. Soc.*, **303** (1987), 193-199.

[2] J. Akeroyd, *A note concerning cyclic vectors in Hardy and Bergman spaces*, Function spaces (Edwardsville, IL, 1990), Lecture Notes in Pure and Appl. Math., vol. 136, pp. 1-8, Dekker, New York, 1992.

[3] J. Akeroyd and K. Alhami, A note on cyclic vectors for the shift, *Complex Var. Theory Appl.*, **48** (2003), 221-224.

[4] J. Akeroyd, D. Khavinson and H.S. Shapiro, Remarks concerning cyclic vectors in Hardy and Bergman spaces, *Michigan Math. J.*, **38** (1991), 191-205.

[5] H. Bercovici, *Factorization theorems for integrable functions*, Analysis at Urbana, Vol. II (Urbana, IL, 1986-1987), London Math. Soc. Lecture Note Ser., vol. 138, pp. 9-21, Cambridge Univ. Press, Cambridge, 1989.

[6] J. Bram, Subnormal operators, *Duke Math. J.*, **22** (1955), 75-94.

[7] S. Brown, C_{00} contractions with spectral boundary, *Integral Equations Operator Theory*, **11** (1988), 49-63.

[8] P.S. Bourdon, Density of the polynomials in Bergman spaces, *Pacific J. Math.*, **130** (1987), 215-221.

[9] J.B. Conway, *The Theory of Subnormal Operators*, Pitman, Boston MA, 1991.

[10] P. Hartman and A. Wintner, On the spectra of Toeplitz's matrices, *Amer. J. Math.*, **72** (1950), 359-366.

[11] G.E. Keough, Subnormal operators, Toeplitz operators and spectral inclusion, *Trans. Amer. Math. Soc.*, **283** (1981), 125-135.

[12] R. Olin and J. Thomson, Algebras generated by a subnormal operator, *Trans. Amer. Math. Soc.*, **271** (1982), 299-311.

[13] R. Olin and L. Yang, The commutant of multiplication by z on the closure of polynomials in $L^t(\mu)$, *J. Funct. Anal.*, **134** (1995), 297-320

[14] R. Olin and L. Yang, A subnormal operator and its dual, *Canad. J. Math.*, **48** (1996), 381-396.

[15] J.Z. Qiu, Density of polynomials, *Houston J. Math.*, **21** (1995), 109-118.

[16] C. Sundberg, Exact sequences for generalized Toeplitz operators, *Proc. Amer. Math. Soc.*, **101** (1987), 634-637.

[17] J. Thomson, Approximation in the mean by polynomials, *Ann. of Math. (2)*, **133** (1991), 477-507.

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