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On the definition of Gelfand-Shilov spaces

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To Professor Ion Colojoară on the occasion of his 80th birthday

Abstract - Gelfand-Shilov spaces can be defined using either sequences of real positive numbers, either the functions associated to these sequences of numbers (considered as weight functions). In this paper we provide a characterization of weight functions which can be used for the definition of Gelfand-Shilov spaces.

Key words and phrases : Gelfand-Shilov space, weight function.

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1. Introduction

In this paper we consider only functions which depend on one real variable. Our definitions and results can be easily extended to functions which depend on more variables.

The Gelfand-Shilov spaces are spaces of rapidly decreasing functions with controlled decrease (of the functions and of their derivatives). They were thoroughly studied in [11] and were used for the study of the Cauchy problem for partial differential equations and systems of partial differential equations.

The original definition of the Gelfand-Shilov spaces is the following. For $(M_p)_p$ and $(N_p)_p$ two sequences of increasing positive numbers and A and B two positive constants one defines

$$\mathcal{S}_{A,B}(\{M_p\},\{N_p\}) =$$

$$= \{\varphi \in \mathcal{S} \,|\, \text{there exists } C > 0 \text{ such that} \sup_{x\in\mathbb{R}} \left| x^q \varphi^{(p)}(x) \right| < CA^p B^q M_p N_q,$$

$$\text{for all } p, q \in N\},$$

where $\mathcal{S}$ is the Schwartz space of rapidly decreasing functions. The Gelfand-Shilov space of Roumieu type, $\mathcal{S}(\{M_p\},\{N_p\})$ ([11], [24]), is the union of these spaces endowed with the inductive limit topology and the Gelfand-Shilov space of Beurling type, $\mathcal{S}((M_p),(N_p))$ ([4], [23], [25]), is the intersection of these spaces, endowed with the projective limit topology.

The most important Gelfand-Shilov spaces are obtained for $M_p = p^{p\alpha}$, $N_p = p^{p\beta}$, $(\forall)p > 0$, with α and β positive constants. The Gelfand-Shilov spaces of Roumieu type defined by these sequences were denoted in [11] with $\mathcal{S}^{\alpha}_{\beta}$.

Many papers were devoted to their study ([4], [6], [10], [15], [20], [22], [25]) and also their range of applications, related especially to the theory of differential and pseudodifferential operators ([5], [14], [23]), increased. The Gelfand-Shilov spaces were also considered from the point of view of time-frequency analysis ([6], [15], [22]) and they were used for the definition of modulation spaces ([7], [15], [21]).

For some of these applications it is more convenient to use instead of sequences of numbers their associated functions. Let us recall that for a logarithmically convex sequence $(M_p)_p$, i.e. a sequence which satisfies

$$M_p^2 \leq M_{p-1}M_{p+1},\ (\forall)p \geq 1, \tag{1.1}$$

and such that $M_0 = 1$, its associated function is defined by the formula

$$M(r) = \sup_{p\geq 0}(p \ln r - \ln M_p),\ (\forall)r > 0. \tag{1.2}$$

Using associated functions as weight functions one can define spaces of rapidly decreasing functions, equal with the Gelfand-Shilov spaces. Many authors define weight functions as being the exponentials of the associated functions. But it is more convenient for us to use the term weight functions for the associated functions of a sequence of numbers. All the weight functions we consider below will be non-negative functions defined on $[0,\infty)$. For associated functions to a logarithmically convex sequence, which are defined only for $r > 0$, we put $M(0) = 0$.

In the second section of the paper we shall shortly examine the different possibilities of defining Gelfand-Shilov spaces.

In the third section we shall introduce a class of weight functions which can be used to define Gelfand-Shilov spaces $\mathcal{S}^{\alpha}_{\beta}$ with $\alpha,\beta > 1/2$. The classes of weight functions we found in the mathematical literature cover only the case of spaces $\mathcal{S}^{\alpha}_{\beta}$ with $\alpha,\beta \geq 1$.

2. Equivalent definitions of Gelfand-Shilov spaces

For simplicity, we shall always assume that the sequences of real numbers $(M_p)_p$ and $(N_p)_p$ satisfy $M_0 = N_0 = 1,\ M_1 \geq 1,\ N_1 \geq 1$. This is not a restriction, since only the behaviour of the sequences at infinity matters.

Gelfand and Shilov remarked in [11] that if a function φ belongs to $\mathcal{S}_{\beta}^{\alpha}$, then there exist positive constants C, A and b such that

$$\left|\varphi^{(p)}(x)\right| \leq CA^p p^{p\alpha} \exp(-b\,|x|^{1/\beta})\,,\ (\forall)x \in \mathbb{R}.$$

Later, Roumieu ([24]) proved that if the sequences $(M_p)_p$ and $(N_p)_p$ are logarithmically convex, then a function φ belongs to $\mathcal{S}(\{M_p\},\{N_p\})$ if and only if there exist positive constants C, A and b such that

$$\left|\varphi^{(p)}(x)\right| \leq CA^p M_p \exp(-N(b\,|x|)\,),\ (\forall)x \in \mathbb{R}.$$

Here N is the function associated to the sequence $(N_p)_p$.

Let us observe that the function associated to the sequence $(p^{p\beta})_p$ is equivalent with $|x|^{1/\beta}$. We say that two weight functions M and N are *equivalent* if there exists a positive constant c such that $M(c^{-1}r) \leq N(r) \leq M(cr),\ (\forall)r \geq 0$.

The next step was done by J. Chung, S.-Y. Chung and D. Kim. They proved in [10] that if the sequences $(M_p)_p$ and $(N_p)_p$ are logarithmically convex and satisfy also the following two assumptions:

i) there exists a constant $H \geq 1$ so that

$$M_{p+q} \leq H^{p+q} M_p M_q,\ \ N_{p+q} \leq H^{p+q} N_p N_q,\ \ (\forall)p,q \geq 0 \qquad (2.1)$$

(Komatsu's ultradifferentibility condition),

ii) there exists a constant $h > 0$ so that

$$M_p \geq (h\sqrt{p})^p\,,\ N_p \geq (h\sqrt{p})^p\ ,\ (\forall)p \geq 0 \qquad (2.2)$$

(the nontriviality condition),
then a function φ belongs to $\mathcal{S}(\{M_p\},\{N_p\})$ if and only if there exist two positive constants A and b such that

$$\sup_x |\varphi(x)| \exp(N(b\,|x|)) < \infty,\ \sup_\xi |\hat{\varphi}(\xi)| \exp(M(a\,|\xi|)) < \infty, \qquad (2.3)$$

where M and N are the functions associated to $(M_p)_p$, respectively $(N_p)_p$ and, as usual, we denoted with $\hat{\varphi}$ the Fourier transform of φ.

A similar characterization holds for Gelfand-Shilov spaces of Beurling type. If the sequences $(M_p)_p$ and $(N_p)_p$ are logarithmically convex, satisfy (2.1) and for every constant $h > 0$ there exists a positive constant c so that

$$M_p \geq c\,(h\sqrt{p})^p\,,\ N_p \geq c\,(h\sqrt{p})^p\ ,\ (\forall)p \geq 0\,, \qquad (2.2')$$

then a function φ belongs to $\mathcal{S}((M_p),(N_p))$ if and only if (2.3) holds for every positive constants a and b.

So, another possibility arises: to define spaces of rapidly decreasing functions using weight functions.

For M and N two weight functions and a and b two positive constants, let us put

$$\mathcal{S}_{a,b}(M,N) =$$
$$= \{\varphi \in \mathcal{S};\ \sup_x |\varphi(x)| \exp(N(b\,|x|)) < \infty,\ \sup_\xi |\hat{\varphi}(\xi)| \exp(M(a\,|\xi|)) < \infty\},$$

and

$$\mathcal{S}(\{M\},\{N\}) = \liminf_{a,b\to 0} \mathcal{S}_{a,b}(M,N),\ \ \mathcal{S}((M),(N)) = \lim_{a,b\to\infty}\mathrm{proj}\, \mathcal{S}_{a,b}(M,N).$$

In the next section we shall give an answer to the following question: what conditions should satisfy the weight functions M and N in order that the spaces $\mathcal{S}(\{M\},\{N\})$ and $\mathcal{S}((M),(N))$ are Gelfand-Shilov spaces?

3. A class of weight functions

Definition 3.1. *A nondecreasing function $M : [0,\infty) \to [0,\infty)$ is called Gelfand-Shilov admissible if*

a) $M(r) = 0,\ (\forall) r \in [0,1]$;

b) there exists $\alpha \geq 1$ such that $2M(r) \leq M(\alpha r),\ (\forall) r \geq 0$;

c) $M \circ \exp$ is a convex function;

d) there exist two positive constants C and β, $\beta < 2$, such that $M(r) \leq Cr^\beta,\ (\forall) r \geq 0$.

Remark 3.1. The first condition from Definition 3.1 has only a technical character. Only the behaviour at infinity of the function M is important for the definition of the spaces $\mathcal{S}(\{M\},\{N\})$ and $\mathcal{S}((M),(N))$.

Definition 3.2. *If M is an admissible Gelfand-Shilov function the sequence $(M_p)_p$ defined by $\ln M_p = \sup_{r>0}(p \ln r - M(r)),\ (\forall) p \in N$ is called the sequence associated to the function M.*

Proposition 3.1. *If M is a Gelfand-Shilov admissible function and $(M_p)_p$ is its associated sequence, then $M_0 = 1$, $(M_p)_p$ is logarithmically convex, satisfies* (2.1) *and there exist $\eta > 1/2$, $h > 0$ such that*

$$M_p \geq h^p p^{p\eta}\,,\ \ (\forall) p \geq 0\,. \tag{3.1}$$

Proof. The fact that $M_0 = 1$ follows immediately from a). The proof of logarithmic convexity is also almost trivial. For $p > 0$ we have

$$\ln M_{p+1} + \ln M_{p-1} = \sup_{r>0}[(p+1)\ln r - M(r)] + \sup_{r>0}[(p-1)\ln r - M(r)] \geq$$

$$\geq \sup_{r>0}[2p\ln r - 2M(r)] = 2\sup_{r>0}[p\ln r - M(r)] = 2\ln M_p.$$

Therefore, $M_p^2 \leq M_{p-1}M_{p+1},\ (\forall)p \geq 1$.

Let us prove now that $(M_p)_p$ satisfies Komatsu's ultradifferentiability condition. We shall use the fact that M satisfies condition b) from Definition 3.1:

$$\ln M_{p+q} = \sup_{r>0}[p\ln r + q\ln r - M(r)] \leq \sup_{r>0}\left[p\ln r + q\ln r - 2M\left(\frac{r}{\alpha}\right)\right] =$$

$$= \sup_{r>0}\left[p\ln\left(\frac{r}{\alpha}\right) + p\ln\alpha + q\ln\left(\frac{r}{\alpha}\right) + q\ln\alpha - 2M\left(\frac{r}{\alpha}\right)\right] \leq$$

$$\leq \ln\alpha^{p+q} + \sup_{r>0}\left[p\ln\left(\frac{r}{\alpha}\right) - M\left(\frac{r}{\alpha}\right)\right] + \sup_{r>0}\left[q\ln\left(\frac{r}{\alpha}\right) - M\left(\frac{r}{\alpha}\right)\right] =$$

$$= \ln\alpha^{p+q} + \ln M_p + \ln M_q = \ln\left(\alpha^{p+q}M_pM_q\right).$$

Finally, let us remark that it is sufficient to prove (3.1) for p sufficiently large. We put

$$\tilde{M}(p) = \sup_{r>0}(p\ln r - M(r)) = \sup_{r\geq 1}(p\ln r - M(r)) = \sup_{t\geq 0}(pt - M(\mathrm{e}^t)),\ (\forall)p \geq 0$$

(the function $\tilde{M}$ is the Young transform of $M \circ \exp$). Then

$$\tilde{M}(p) \geq \sup_{t\geq 0}(pt - C\mathrm{e}^{\beta t}) = \frac{p}{\beta}(\ln p - \ln\beta C - 1),\ (\forall)p > \beta C.$$

Hence, for every $\delta > 0$ there exists $p_\delta > \beta C$ so that

$$\tilde{M}(p) \geq \frac{p}{\beta+\delta}\ln p,\ (\forall)p > p_\delta.$$

Since $\beta < 2$, we can choose δ such that $\frac{1}{\beta+\delta} > \frac{1}{2}$. □

Proposition 3.2. *Let $(M_p)_p$ be a sequence of positive numbers. If $M_0 = 1$, $(M_p)_p$ is logarithmically convex and satisfies* (2.1) *and* (3.1)*, then its associated function M is a Gelfand-Shilov admissible weight function.*

Proof. Condition a) and the fact that M is nondecreasing follow straight from the definition of the associated function and $M_0 = 1$. It is also well known that if M is the function associated to a logarithmically convex sequence, then $M \circ \exp$ is a convex function (see [18], [24]). We also proved in [19] (Lemma 5), that if (2.1) and (3.1) hold for the sequence $(M_p)_p$, then

$$M(br) + M(cr) \leq M(ar),\ (\forall) b, c, r > 0,$$

with $a = H \max(b, c)$.

If $b = c = 1$, one obtains b) from Definition 3.1.

Finally, if $(M_p)_p$ satisfies (3.1), then

$$\sup_p (p \ln r - \ln M_p) \leq \sup_p (p \ln r - p \ln h - p\eta \ln p) =$$

$$= \sup_p \left(p \ln \frac{r}{h} - p\eta \ln p \right) \leq C \left(\frac{r}{he} \right)^{1/\eta},\ (\forall) r > 0$$

for some positive constant C. □

Lemma 3.1. *If M is a Gelfand-Shilov admissible function, then there exists two positive constants c and q such that $M(r) \geq cr^q$, $(\forall) r \geq 0$.*

Proof. We use property b):

$$M(r) \geq 2^n M \left(\frac{r}{\alpha^n} \right),\ (\forall) n \in \mathbb{N}^*.$$

If we take $n = \left[\frac{\ln r}{\ln \alpha} \right]$, we see that $M(r) \geq \frac{1}{2} M(1)\, r^{\frac{\ln 2}{\ln \alpha}}$. So $M(r) \geq cr^q$, $(\forall) r \geq 0$, for $c = \frac{M(1)}{2}$ and $q = \frac{\ln 2}{\ln \alpha}$. □

Proposition 3.3. *If M is a Gelfand-Shilov admissible function, $(M_p)_p$ is its associated sequence and $\bar{M}$ is the function associated to $(M_p)_p$, then $\bar{M}$ is equivalent with M.*

Proof. Accordingly to Lemma 3.1, $\lim\limits_{t \to \infty} \frac{M(e^t)}{t} = \infty$. Therefore $M \circ \exp$ coincides with the Young transform of its Young transform $\tilde{M}$:

$$M(r) = \sup_{p \geq 0} (p \ln r - \tilde{M}(p)),\ (\forall) r > 0.$$

We must prove that M is equivalent with

$$\bar{M}(r) = \sup_{p \in \mathbb{N}} (p \ln r - \ln M_p),\ (\forall) r > 0,$$

where

$$\ln M_p = \sup_{r > 0} (p \ln r - M(r)),\ (\forall) p \in \mathbb{N}.$$

Since $\tilde{M}(p) = \ln M_p,\ (\forall)p \in \mathbb{N}$, it is clear that $\bar{M}(r) \leq M(r),\ (\forall)r > 0$.

On the other hand, let us first remark that

$$\ln M_{q+1} = \sup_{r>0}[(q+1)\ln r - M(r)] \leq \sup_{r>0}\left[(q+1)\ln r - 2M\left(\frac{r}{\alpha}\right)\right] \leq$$

$$\leq \sup_{r>0}\left[q\ln r - M\left(\frac{r}{\alpha}\right)\right] + \sup_{r>0}\left[\ln r - M\left(\frac{r}{\alpha}\right)\right] \leq$$

$$\leq \sup_{r>0}\left[q\ln r - M\left(\frac{r}{\alpha}\right)\right] + \sup_{r>0}\left[\ln r - c\left(\frac{r}{\alpha}\right)^q\right] \leq$$

$$\leq \sup_{r>0}[q\ln(\alpha r) - M(r)] + C = C + q\ln\alpha + \ln M_q.$$

So, there exists some positive constant γ such that

$$\ln M_{q+1} \leq (q+1)\ln\gamma + \ln M_q.$$

For p a real positive number, we take $q = [p]$. Since $\tilde{M}$ is an increasing function, we get

$$\ln M_{[p]+1} \leq ([p]+1)\ln\gamma + \tilde{M}(p).$$

Therefore

$$p\ln r - \tilde{M}(p) \leq ([p]+1)\ln\gamma + ([p]+1)\ln r - \ln M_{[p]+1}.$$

Hence

$$M(r) = \sup_{p\geq 0}(p\ln r - \tilde{M}(p)) \leq \sup_{p\in N}(p\ln\gamma r - \ln M_p) = \bar{M}(\gamma r),\ (\forall)r > 0.$$

□

Remark 3.2. If M is equivalent with $\bar{M}$ and N is equivalent with $\bar{N}$, then

$$\mathcal{S}(\{M\},\{N\}) = \mathcal{S}(\{\bar{M}\},\{\bar{N}\}) \text{ and } \mathcal{S}((M),(N)) = \mathcal{S}((\bar{M}),(\bar{N})).$$

Summing up, we proved the following theorem.

Theorem 3.1. *The spaces* $\mathcal{S}(\{M\},\{N\})$ and $\mathcal{S}((M),(N))$ *can be realized as Gelfand-Shilov spaces* $\mathcal{S}(\{M_p\},\{N_p\})$, *respectively* $\mathcal{S}((M_p),(N_p))$, *with* $(M_p)_p$ *and* $(N_p)_p$ *logarithmically convex sequences,* $M_0 = N_0 = 1$ *and satisfying* (2.1) *and* (3.1) *if and only if* M *and* N *are Gelfand-Shilov admissible weight functions.*

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