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A selection of embedding results for Lipschitz manifolds

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To Professor Ion Colojoară on the occasion of his 80th birthday

Abstract - The present paper is a selection of results concerning embedding theorems for Lipschitz manifolds that aims to emphasize the contribution of Professor Ion Colojoară to this topic.

Key words and phrases : Lipschitz manifolds, embedding.

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1. Preliminaries

The basic results needed to do Lipschitz analysis on metric spaces could be found in the seminal paper [2] of J. Luukkainen and J. Väisälä.

Definition 1.1. *Given two metric spaces* (X, d) *and* $(X^{'}, d^{'})$*, a function* $f : X \to X^{'}$ *is called Lipschitz if there exists a constant* L *such that*

$$d^{'}(f(x), f(y)) \leq Ld(x, y),$$

for all $x, y \in X$.

If every $x \in X$ *has a neighborhood* U *such that* $f_{|U}$ *is Lipschitz,* f *is called locally Lipschitz.*

Definition 1.2. *Given two metric spaces* (X, d) *and* $(X^{'}, d^{'})$*, a function* $f : X \to X^{'}$ *is called a lipeomorphism if it is a bijection such that* f *and* f^{-1} *are locally Lipschitz; it is called a LIP embedding if it is injective and* f *defines a lipeomorphism* $f_1 : X \to f(X)$*; it is called a LIP immersion if every* $x \in X$ *has a neighborhood* U *such that* $f_{|U}$ *is a LIP embedding.*

On one hand, taking into account the classical theorem of H. Rademacher which states that a Lipschitz function $f : U = \overset{\circ}{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable outside of a Lebesgue null set of U, the condition of being Lipschitz could be viewed as a weakened version of differentiability. On the other hand, from the intuitive point of view one can imagine a Lipschitz function as one that obeys temporary speed limits.

2. Lipschitz manifolds

A Lipschitz manifold (a manifold with corners) is a topological manifold with an extra structure which on one hand is slightly weaker than a smooth structure (and therefore one can do analysis with it) and on the other hand the essential uniqueness of it is automatic in many situations.

Let us present, following [1], the definition of a Lipschitz manifold.

Definition 2.1. *A Lipschitz B-manifold, where B is a Banach space, is a Hausdorff topological space X equipped with a family of charts $h_\alpha : U_\alpha \to B$, satisfying the following conditions:*

i) the family $(U_\alpha)_{\alpha \in A}$ is an open cover of X;
ii) each h_α is a homeomorphism onto the open subset $h_\alpha(U_\alpha)$ of B;
iii) the change of coordinates $h_\beta \circ h_\alpha^{-1} : h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\alpha \cap U_\beta)$ is locally Lipschitz, for all $\alpha, \beta \in A$.

Remark 2.1. Let us note that $h_\beta \circ h_\alpha^{-1} : h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\alpha \cap U_\beta)$ is a lipeomorphism, for all $\alpha, \beta \in A$.

Remark 2.2. A Lipschitz $\mathbb{R}^n$-manifold is called a Lipschitz manifold of dimension n or a Lipschitz n-manifold. Basically the same definition as above (with the supplementary condition that X is a second countable locally compact space), for finite dimensional Lipschitz manifolds, can be found in [4], page 270 or [2], page 97-98.

J. Luukkainen and J. Väisälä (see [2]) prefer to use the following alternative definition of a Lipschitz n-manifold which is basically equivalent with the above one but conceptually simpler.

Definition 2.2. *A Lipschitz n-manifold is a separable metric space X such that every point $x \in X$ has a closed neighborhood U lipeomorphic to $[-1, 1]^n$.*

N. Teleman (see [5]) shows how to do analysis on finite dimensional compact connected Lipschitz manifolds so that the signature operators can be defined. Actually Teleman generalizes the Atiyah-Singer index theorem to closed topological oriented manifolds which admit a Lipschitz structure. The feasibility of such an approach is based on the famous Sullivan's theorem on the existence of an essentially unique finite dimensional Lipschitz structure on every compact topological manifold of dimension not equal to 4. According to [4], Freedman, Donaldson and others proved that there are topological 4-manifolds with no Lipschitz structure.

3. Embedding results for Lipschitz manifolds

Although the main research area of professor Ion Colojoară is spectral theory, he made important contributions to the embedding theory of infinite dimensional manifolds.

According to the famous Whitney's embedding theorem, every C^∞ manifold of dimension n can be C^∞ embedded in $\mathbb{R}^{2n+1}$.

In 1965 professor Colojoară, in the same time with J.H. McAlpin, gave a generalization of this result. Namely he proved that every paracompact second countable C^∞ manifold modelled on a separable Hilbert space H admits a C^∞ embedding into H.

In 1977 J. Luukkainen and J. Väisälä proved the following embedding result for finite dimensional Lipschitz manifolds.

Theorem 3.1. *If X is a second countable paracompact Lipschitz n-manifold, then there exists an injective function $f : X \to \mathbb{R}^{n(n+1)}$ such that f and $f^{-1} : f(X) \to X$ are locally Lipschitz and $f(X)$ is closed.*

In 1995 professor Colojoară returned to the subject of embedding of infinite dimensional manifolds, providing an embedding theorem for second countable paracompact Lipschitz manifolds modeled on separable Hilbert spaces. More precisely, he proved the following result (see [1]):

Theorem 3.2. *If X is a paracompact second countable H-Lipschitz manifold, where H is a separable Hilbert space, there exists an injective function $h : X \to H$ such that h and $h^{-1} : f(X) \to X$ are locally Lipschitz and $h(X)$ is closed.*

The idea of the proof is the following: let us consider a countable H-Lipschitz atlas $\{h_i : G_i \to H\}_{i\in\mathbb{N}}$ and two locally finite open covers of X, namely $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ such that $\overline{V_i} \subseteq U_i \subseteq \overline{U_i} \subseteq G_i$, for all $i \in \mathbb{N}$.

Then, a result from [2] assures us that, for every $i \in \mathbb{N}$, there exists a locally Lipschitz function $f_i : X \to [0, 1]$ such that $\operatorname{supp}(f_i) \subseteq U_i$, $f_i(x) = 1$, for all $x \in \overline{V_i}$, and $f_i(x) < 1$, for all $x \notin \overline{V_i}$.

Let us note that $H \simeq l^2$.

We can consider the locally Lipschitz functions $f, F : X \to l^2$, $g_i : X \to H$, $g : X \to \oplus^{\mathbb{N}} H$ and $h : X \to H \oplus \oplus^{\mathbb{N}} H$ given by

$$f(x) = (\frac{f_1(x)}{2}, \frac{f_2(x)}{2^2}, ..., \frac{f_i(x)}{2^i}, ..),$$

$$F(x) = \frac{f(x)}{\|f(x)\|^2},$$

$$g_i(x) = \{ \begin{array}{cc} f_i(x)h_i(x), & x \in U_i \\ 0, & x \notin U_i \end{array},$$

$$g(x) = (g_1(x), g_2(x), ..., g_i(x), ...)$$

and

$$h(x) = (F(x), g(x))$$

for every $x \in X$ and $i \in \mathbb{N}$.

Checking that the function $h : X \to H$ is injective, h and $h^{-1} : f(X) \to X$ are locally Lipschitz and $h(X)$ is closed, together with the existence of the following unitary isomorphisms $H \oplus \oplus^{\mathbb{N}} H \simeq H \oplus H \simeq H$, will finish the proof.

Finally let us mention that in [3] it is proved that for each paracompact and second countable B-Lipschitz manifold, where B could be l^p, L^p, $p \in (1, \infty)$, c_0 or $c_0 \oplus l^2$, there exists a continuos and injective function $f : X \to B$ having the property that every point $x \in X$ has a neighborhood U such that $f_{|U} : U \to f(U)$ is a lipeomorphism. Therefore f is a LIP immersion.

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