Annals of the University of Bucharest (mathematical series)
(Analele Universităţii Bucureşti. Matematică)
1 (LIX) (2010), 111–119

Compact magnetic pseudodifferential operators

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To Professor Ion Colojoară on the occasion of his 80th birthday

Abstract - In previous papers, a generalisation of the Weyl calculus was introduced and studied, in connection with the quantization of a particle moving in $\mathbb{R}^d$ under the influence of a variable magnetic field B. In the present article we prove a criterion for the corresponding magnetic pseudodifferential operators to be compact. We apply this criterion to the study of the parametrix of an elliptic operator.

Key words and phrases : magnetic pseudodifferential operators, compactness, elliptic operators, parametrix.

Mathematics Subject Classification (2000) **:** 35S05, 47G30, 81Q99.

1. Introduction

Let B be a magnetic field, i.e. a closed 2-form on $\mathbb{R}^d$ with components $B_{jk}(1 \leq j, k \leq d)$ of class $BC^\infty(\mathbb{R}^d)$; it can be written as the differential $\mathrm{d}A$ of a 1-form A on $\mathbb{R}^d$ with components $A_j(1 \leq j \leq d)$ of class $\mathcal{C}^\infty(\mathbb{R}^d)$, for which all the derivatives have polynomial growth. In a series of papers (see [3], [4], [5], [1], [2]) a 'magnetic pseudodifferential calculus' was proposed; this is a gauge covariant functional calculus (which is the Weyl calculus if $B = 0$), i.e. a systematic procedure to associate to suitable 'classical observables' f (usually f belongs to Hörmander's symbol classes $S^m(\mathbb{R}^d), m \in \mathbb{R}$) the operators $\mathcal{O}_p^A(f) \in \mathcal{B}(\mathcal{S}(\mathbb{R}^d))$, defined by oscillatory integrals:

$$[\mathcal{O}_p^A(f)u](y) := \int_{\mathbb{R}^{2d}} e^{i(\langle x-y,\eta\rangle - \Gamma^A(x,y))} f\left(\frac{x+y}{2}, \eta\right) u(y)\mathrm{d}y\bar{\mathrm{d}}\eta, \tag{1.1}$$

where $\bar{\mathrm{d}}\eta := (2\pi)^{-d}\mathrm{d}\eta, u \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d$ and $\Gamma^A(x,y) := \int_{[x,y]} A$ is the circulation of A along the segment $[x,y]$.

We use the notations and results of [1], where this 'magnetic pseudodifferential calculus' was developed.

Let, for every $s \in \mathbb{R}$, $\mathcal{H}_A^s$ be the magnetic Sobolev space defined in [1]; for every $m, s \in \mathbb{R}$ and $f \in S^m(\mathbb{R}^d)$ we have $\mathcal{O}_p^A(f) \in \mathcal{B}(\mathcal{H}_A^s, \mathcal{H}_A^{s-m})$. The first main result of this paper consists in the following theorem

Theorem 1.1. *Let $s, t, m \in \mathbb{R}$ and $f \in S^m(\mathbb{R}^d)$. We can choose two positive integers N', N'' depending on s, t, m such that if the following hypothesis hold*

i) $m < s - t$;

ii) we have

$$\lim_{|x|\to\infty} \partial_x^\alpha \partial_\xi^\beta f(x, \xi) = 0, \ \forall \xi \in \mathbb{R}^d, \alpha, \beta \in \mathbb{N}^d, |\alpha| \leq N', |\beta| \leq N'', \quad (1.2)$$

then the operator $\mathcal{O}_p^A(f) : \mathcal{H}_A^s \to \mathcal{H}_A^t$ is compact.

Let $f \in S^m(\mathbb{R}^d)$ be an elliptic symbol. As we know from [1], there exists $g \in S^{-m}(\mathbb{R}^d)$ such that

$$a := f \#^B g - 1 \in S^{-\infty}(\mathbb{R}^d), b := g \#^B f - 1 \in S^{-\infty}(\mathbb{R}^d), \quad (1.3)$$

where $f \#^B g$ stands for the symbol of the composition $\mathcal{O}_p^A(f) \circ \mathcal{O}_p^A(g)$. $\mathcal{O}_p^A(g)$ is an approximate inverse for $\mathcal{O}p^A(f)$, called parametrix. Generally the operators $\mathcal{O}_p^A(a)$ and $\mathcal{O}_p^A(b)$ are smoothing, but not compact. The next main result is given by

Theorem 1.2. *Let $f \in S^m(\mathbb{R}^d)$. Suppose that there exists a positive constant c such that*

$$|f(x, \xi)| \geq c \langle \xi \rangle^m, \ \textit{for all } x, \xi \in \mathbb{R}^d. \quad (1.4)$$

Then we can choose g, a, b in (1.3) such that for all $s, t \in \mathbb{R}$, there exist the positive integers $\overline{N}, \overline{N}', \overline{N}''$ (depending on s, t, m), such that if the following assumptions hold

$$\lim_{|x|\to\infty} |\partial^\alpha B(x)| = 0, \ \textit{for all } \alpha \in \mathbb{N}^d, |\alpha| \leq \overline{N} \quad (1.5)$$

and

$$\lim_{|x|\to\infty} \partial_x^\alpha \partial_\xi^\beta f(x, \xi) = 0, \ \forall \xi \in \mathbb{R}^d, \alpha, \beta \in \mathbb{N}^d, 1 \leq |\alpha| \leq \overline{N}', |\beta| \leq \overline{N}'', \quad (1.6)$$

then the operators $\mathcal{O}_p^A(a) : \mathcal{H}_A^s \to \mathcal{H}_A^t$ and $\mathcal{O}_p^A(b) : \mathcal{H}_A^s \to \mathcal{H}_A^t$ are compact.

In section 2 we study the boundedness (a result of Calderon and Vaillancourt type) and compactness of a kind of operators more general than the one defined in (1.1), which implies the Theorem 1.1 in the case $s = t = 0$. Section 3 is dedicated to some properties of the magnetic composition. In section 4 we provide the proof of Theorem 1.1 in the general case. The last section is devoted to the study of a parametrix of an elliptic operator under the assumptions of Theorem 1.2.

2. Compactness of magnetic pseudodifferential operators

We are going to investigate the following operator (defined by an oscillatory integral)

$$T_a u(x) := \int_{\mathbb{R}^{2d}} e^{i(\langle x-y,\xi\rangle - \Gamma^A(x,y))} a(x,y,\xi) u(y) \mathrm{d}y \bar{\mathrm{d}}\xi, x \in \mathbb{R}^d, \tag{2.1}$$

where $u \in \mathcal{S}(\mathbb{R}^d)$ and $a \in BC^\infty(\mathbb{R}^{3d})$. We have $T_a \in \mathcal{B}(\mathcal{S}(\mathbb{R}^d))$; in order to prove that $T_a \in \mathcal{B}(L^2(\mathbb{R}^d))$, we need the following lemma (see [1]).

Lemma 2.1. *Let $F(x,y,z)$ be the flux of the 2-form B through the triangle $[x-y+z, x-y-z, x+y-z]$, that is*

$$F(x,y,z) := \int_{[x-y+z,x-y-z,x+y-z]} B, \ \ x,y,z \in \mathbb{R}^d. \tag{2.2}$$

Then $\nabla_x F, \nabla_y F$ and $\nabla_z F$ are of the form $D(x,y,z)y + E(x,y,z)z$, where D and E are $d \times d$ antisymmetrical matrices with components of class $BC^\infty(\mathbb{R}^d)$. In addition, if

$$\lim_{|x|\to\infty} \partial^\alpha B_{jk}(x) = 0, \ \text{for all} \ \ 1 \le j,k \le d,$$

for an $\alpha \in \mathbb{N}^d$, then

$$\lim_{|x|\to\infty} (|\partial^\alpha D(x,y,z)| + |\partial^\alpha E(x,y,z)|) = 0, \ \text{for all} \ \ y,z \in \mathbb{R}^d.$$

Proposition 2.1. *In the conditions above $T_a \in \mathcal{B}(L^2(\mathbb{R}^d))$ and*

$$\|T_a\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \le M := C \sup_{x,y,\xi\in\mathbb{R}^d, |\alpha|\le 2d+2, |\beta|\le 3d+4} |\partial_y^\alpha \partial_\xi^\beta a(x,y,\xi)|, \tag{2.3}$$

where C is a positive constant depending only on d.

Proof. The proof is quite standard. Choose $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{3d})$ with $\chi(0,0,0) = 1$ and for $\varepsilon \in [0,1]$ define $a_\varepsilon(x,y,\xi) := \chi(\varepsilon x, \varepsilon y, \varepsilon\xi) a(x,y,\xi)$. It holds $\lim_{\varepsilon\searrow 0} K_{a_\varepsilon} = K_a$ in $\mathcal{S}'(\mathbb{R}^{2d})$, where K_{a_ε} stands for the distribution kernel of the operator $T_{a_\varepsilon}, 0 \le \varepsilon \le 1$. The derivatives of a_ε are estimated via the derivatives of a, uniformly with respect to ε, and therefore it is sufficient to prove the estimate (2.3) for $a \in \mathcal{S}(\mathbb{R}^{3d})$.

Using the operator $\langle x-y\rangle^{-2p}(1-\Delta_\xi)^p$, with $p = \left[\frac{3d}{2}\right] + 2$ and integrating by parts we get

$$T_a u(x) = \int_{\mathbb{R}^d} (P_\xi u)(x) \bar{\mathrm{d}}\xi, \ u \in \mathcal{S}(\mathbb{R}^d), \ x,\xi \in \mathbb{R}^d. \tag{2.4}$$

Here P_ξ is an integral operator with the integral kernel

$$P_\xi(x,y) := e^{i(\langle x-y,\xi\rangle - \Gamma^A(x,y))} b(x,y,\xi),$$

$$b(x,y,\xi) := \langle x-y\rangle^{-2p}(1-\Delta_\xi)^p a(x,y,\xi),\ x,y,\xi \in \mathbb{R}^d.$$

It is obvious that $P_\xi \in \mathcal{B}(L^2(\mathbb{R}^d))$. We use now the Cotlar-Knapp-Stein lemma in order to prove that there exists a function $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that the following inequalities hold

$$\|P_\xi P_\eta^*\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \le h^2(\xi,\eta), \tag{2.5}$$

$$\|P_\xi^* P_\eta\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \le h^2(\xi,\eta), \tag{2.6}$$

for $\xi, \eta \in \mathbb{R}^d$ and

$$\|H\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \le M, \tag{2.7}$$

where H is the integral operator with the integral kernel h.

The Stokes formula and an integration by parts with the operator given by $\langle \xi-\eta\rangle^{-2q}(1-\Delta_z)^q$, $q = d+1$ allow us to see that the integral kernel of the operator $P_\xi P_\eta^*$ is given by

$$K_{\xi,\eta}(x,y) := e^{i(\langle x,\xi\rangle - \langle y,\eta\rangle - \Gamma^A(x,y))}\cdot \int_{\mathbb{R}^d} e^{i\langle z,\eta-\xi\rangle}\langle \xi-\eta\rangle^{-2q}(1-\Delta_z)^q\left[e^{-iG(x,y,z)}b(x,z,\xi)\overline{b(y,z,\eta)}\right]\mathrm{d}z, \tag{2.8}$$

for $x,y,\xi,\eta \in \mathbb{R}^d$, where $G(x,y,z) := F\left(\frac{x+z}{2}, \frac{z-y}{2}, \frac{x-y}{2}\right)$. Using Lemma 2.1, we notice that

$$\int_{\mathbb{R}^d} |K_{\xi,\eta}(x,y)|dy \le h^2(\xi,\eta),\ \int_{\mathbb{R}^d} |K_{\xi,\eta}(x,y)|dx \le h^2(\xi,\eta),\ x,y,\xi,\eta \in \mathbb{R}^d, \tag{2.9}$$

with $h(\xi,\eta) := M\langle \xi-\eta\rangle^{-q}$, implying the estimates (2.5), (2.6) and (2.7). □

The next result shows that under an additional assumption imposed on a the operator T_a is compact.

Proposition 2.2. *Let $a \in BC^\infty(\mathbb{R}^{3d})$. Assume that there exists $t \in [0,1]$ such that*

$$\lim_{|tx+(1-t)y|+|\xi|\to\infty} \partial_y^\alpha \partial_\xi^\beta a(x,y,\xi) = 0,\ \forall \alpha,\beta \in \mathbb{N}^d, |\alpha| \le 2d+2, |\beta| \le 3d+4. \tag{2.10}$$

Then T_a is a compact operator on $L^2(\mathbb{R}^d)$.

Proof. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, $\chi(x,\xi) = 1$ for $|x| + |\xi| \le 1$. Set $\chi_R(x,\xi) := \chi\left(\frac{x}{R}, \frac{\xi}{R}\right)$, $R \ge 1$ and $a_R(x,\xi) := \chi_R(tx + (1-t)y, \xi) a(x,y,\xi)$. Using (2.10) it follows

$$\lim_{R\to\infty} \sup_{\mathbb{R}^{3d}} \left| \partial_y^\alpha \partial_\xi^\beta [a_R(x,y,\xi) - a(x,y,\xi)] \right| = 0,$$

for all $\alpha, \beta \in \mathbb{N}^d, |\alpha| \le 2d+2, |\beta| \le 3d+4$. Hence, by virtue of Proposition 2.1, $\lim_{R\to\infty} T_{a_R} = T_a$ in $B(L^2(\mathbb{R}^d))$. Since the distribution kernel of the operator T_{a_R} is a function from $\mathcal{S}(\mathbb{R}^{2d})$, it yields that the operator T_{a_R} is compact in $L^2(\mathbb{R}^d)$. The proof is finished. □

In order to apply this proposition to the operator $O_p^A(f)$ appearing in Theorem 1.1 we need the following elementary lemma

Lemma 2.2. *Let $a \in \mathcal{C}(\mathbb{R}^{2d})$ be a function which satisfies*

i) for each $x \in \mathbb{R}^d$ the function $a(x,\cdot)$ is continuous, uniformly with respect to x;

ii) $\lim_{|x|\to\infty} a(x,\xi) = 0$, for all $\xi \in \mathbb{R}^d$;

iii) $\lim_{|\xi|\to\infty} a(x,\xi) = 0$, uniformly with respect to $x \in \mathbb{R}^d$.

Then $\lim_{|x|+|\xi|\to\infty} a(x,\xi) = 0$.

Corollary 2.1. *Theorem 1.1 holds true in the case $s = t = 0$, with $N' = 2d+2, N'' = 3d+4$.*

Proof. We have $O_p^A(f) = T_a$, where $a(x,y,\xi) := f\left(\frac{x+y}{2}, \xi\right)$, $x, y, \xi \in \mathbb{R}^d$. Then $a \in BC^\infty(\mathbb{R}^{3d})$ if $m < 0$ and the formula (2.10) with $t = \frac{1}{2}$ is a consequence of (1.2), Lemma 2.2 and the fact that $f \in \mathcal{S}^m(\mathbb{R}^d)$, $m < 0$. □

3. Some properties of the magnetic composition

Let f and g be two symbols, $f \in S^m(\mathbb{R}^d)$, $g \in S^{m'}(\mathbb{R}^d)$, $m, m' \in \mathbb{R}$. As we know from [1], $O_p^A(f) \circ O_p^A(g) = O_p^A(f \#^B g)$, where the symbol $f \#^B g \in S^{m+m'}(\mathbb{R}^d)$ is defined by the oscillatory integral

$$(f\#^B g)(X) := \int_{\mathbb{R}^{4d}} e^{-2i[Y,Z]} \omega^B(x,y,z) f(X-Y) g(X-Z) \bar{\mathrm{d}}Y \bar{\mathrm{d}}Z, \qquad (3.1)$$

where $X = (x,\xi)$, $Y = (y,\eta)$, $Z = (z,\zeta)$, $x,y,z,\xi,\eta,\zeta \in \mathbb{R}^d$, $[Y,Z] = \langle \eta, z\rangle - \langle \zeta, y\rangle$, $\bar{\mathrm{d}}Y = \pi^{-n}\mathrm{d}y\mathrm{d}\eta$, $\bar{\mathrm{d}}Z = \pi^{-n}\mathrm{d}z\mathrm{d}\zeta$, $\omega^B(x,y,z) = e^{-iF(x,y,z)}$ (F is defined in Lemma 2.1).

Set $N_1 = N_2 := \left[\frac{d}{2}\right] + 1, N_3 := \left[\frac{d+m_+}{2}\right] + 1, N_4 := \left[\frac{d+m'_+}{2}\right] + 1$, where $m_\pm := \max(\pm m, 0)$.

Proposition 3.1. *Assume that there exist $p, q \in \mathbb{N}$ such that the following hypothesis hold true*

i) $\lim_{|x|\to\infty} |\partial^\alpha B(x)| = 0$, *for all* $\alpha \in \mathbb{N}^d, |\alpha| \le p + 1 + 2(N_3 + N_4)$.

ii) $\lim_{|x|\to\infty} \partial_x^\alpha \partial_\xi^\beta f(x, \xi) = 0$, *for all* $\alpha, \beta \in \mathbb{N}^d, 1 \le |\alpha| \le 2N_4 + 1 + p, |\beta| \le 2N_2 + q$.

iii) $\lim_{|x|\to\infty} \partial_x^\alpha \partial_\xi^\beta g(x, \xi) = 0$, *for all* $\alpha, \beta \in \mathbb{N}^d, 1 \le |\alpha| \le 2N_3 + 1 + p, |\beta| \le 2N_1 + q$.

Then we have the identity

$$f \#^B g = fg + r, \ \ r \in S^{m+m'-1}(\mathbb{R}^d) \tag{3.2}$$

and

$$\lim_{|x|\to\infty} \partial_x^\gamma \partial_\xi^\delta r(x, \xi) = 0, \ \textit{for all}\ \xi \in \mathbb{R}^d, \gamma, \delta \in \mathbb{N}^d, |\gamma| \le p, |\delta| \le q. \tag{3.3}$$

Proof. We use the identity $h(1) = h(0) + \int_0^1 h'(t)\mathrm{d}t$ with $h(t) := f(X - tY)g(X - tZ)$, $X, Y, Z \in \mathbb{R}^{2d}$ and integrate by parts in order to get the equation (3.2), with

$$r(X) = -\frac{1}{2i}\int_0^1 \mathrm{d}t \int_{\mathbb{R}^{4d}} e^{-2i[Y,Z]} L_1 L_2 L_3 L_4 R(t, X, Y, Z) \bar{\mathrm{d}}Y \bar{\mathrm{d}}Z, \tag{3.4}$$

where $L_1 = \langle y\rangle^{-2N_1}(1-\frac{1}{4}\Delta_\zeta)^{N_1}, L_2 = \langle z\rangle^{-2N_2}(1-\frac{1}{4}\Delta_\eta)^{N_2}, L_3 = \langle \eta\rangle^{-2N_3}(1-\frac{1}{4}\Delta_z)^{N_3}, L_4 = \langle \zeta\rangle^{-2N_4}(1-\frac{1}{4}\Delta_y)^{N_4}$ and

$$\begin{aligned} R(t, X, Y, Z) = \omega^B(x, y, z)[&2t\langle(\nabla_x f)(X - tY), (\nabla_\xi g)(X - tZ)\rangle \\ &- 2t\langle(\nabla_\xi f)(X - tY), (\nabla_x g)(X - tZ)\rangle \\ &- i\langle(\nabla_\xi f)(X - tY), (\nabla_z F)(x, y, z)\rangle g(X - tZ) \\ &+ if(X - tY)\langle(\nabla_y F)(x, y, z), (\nabla_\xi g)(X - tZ)\rangle]. \end{aligned}$$

We deduce the formula (3.3) by a careful examination of the integral (3.4), where we use the hypothesis $i) - iii)$, the Lemma 2.1, some additional integrations by parts in order to eliminate the terms of the form $y^\alpha z^\beta, \alpha, \beta \in \mathbb{N}^d$ and the dominated convergence theorem. □

In a similar way we obtain the following result

Proposition 3.2. *Assume that there exist $p, q \in \mathbb{N}$ such that*

i) $\lim_{|x|\to\infty} \partial_x^\alpha \partial_\xi^\beta f(x, \xi) = 0$, *for all* $\alpha, \beta \in \mathbb{N}^d, |\alpha| \le 2N_4 + p, |\beta| \le 2(N_2 + N_3 + N_4) + p + q$

or

i') $\lim_{|x|\to\infty} \partial_x^\alpha \partial_\xi^\beta g(x,\xi) = 0$, *for all* $\alpha, \beta \in \mathbb{N}^d, |\alpha| \le 2N_3 + p, |\beta| \le 2(N_1 + N_3 + N_4) + p + q$.

Then

$$\lim_{|x|\to\infty} \partial_x^\gamma \partial_\xi^\delta (f\#^B g)(x,\xi) = 0, \text{ for all } \xi \in \mathbb{R}^d, \gamma, \delta \in \mathbb{N}^d, |\gamma| \le p, |\delta| \le q. \tag{3.5}$$

4. Compactness in magnetic Sobolev spaces

In this section we give the proof of Theorem 1.1 for arbitrary $s, t \in \mathbb{R}$; in the case $s = t = 0$ this was achieved in Corollary 2.1. Let $\{s_r\}_{r\in\mathbb{R}}$ be the family of symbols considered in [2]; we have $s_r \in \mathcal{S}^r(\mathbb{R}^d)$, $s_0 = 1$ and $s_r\#^B s_{-r} = 1$. Then $a = s_{-t}\#^B b\#^B s_r$, where $b = s_t\#^B a\#^B s_{-r} \in \mathcal{S}^{\overline{m}}(\mathbb{R}^d)$, $\overline{m} := m + t - r < 0$. We have $\mathcal{O}_p^A(s_r) \in \mathcal{B}(\mathcal{H}_A^p, \mathcal{H}_A^{p-r})$, for any $p \in \mathbb{R}$ and the desired conclusion yields if we prove that $O_p^A(b)$ is a compact operator of $L^2(\mathbb{R}^d)$. This fact is a consequence of Corollary 1.1 if we check that

$$\lim_{|x|\to\infty} \partial_x^\alpha \partial_\xi^\beta b(x,\xi) = 0, \ \forall \alpha, \beta \in \mathbb{N}^d, |\alpha| \le N_0' := 2d + 2, |\beta| \le N_0'' := 3d + 4, \tag{4.1}$$

and this will follow from Proposition 3.2, with appropriate choices for N', N''. We have

$$\lim_{|x|\to\infty} \partial_x^\gamma \partial_\xi^\delta (s_t\#^B a)(x,\xi) = 0, \ \forall \gamma, \delta \in \mathbb{N}^d, |\gamma| \le \overline{N}_0', |\beta| \le \overline{N}_0'', \tag{4.2}$$

where $\overline{N}_0' = N' - 2\tilde{N}_3$, $\overline{N}_0'' = N'' - N' - 2(\tilde{N}_1 + \tilde{N}_4)$ and $\tilde{N}_1 = \tilde{N}_2 = \left[\frac{d}{2}\right] + 1$, $\tilde{N}_3 = \left[\frac{d+t_+}{2}\right] + 1, \tilde{N}_4 = \left[\frac{d+m_+}{2}\right] + 1$. Finally, in order to get (4.1), we need $N_0' = \overline{N}_0' - 2\underline{N}_4$, $N_0'' = \overline{N}_0'' - \overline{N}_0' - 2(\underline{N}_2 + \underline{N}_3)$, where $\underline{N}_1 = \underline{N}_2 = \left[\frac{d}{2}\right] + 1, \underline{N}_3 = \left[\frac{d+(m+t)_+}{2}\right] + 1, \underline{N}_4 = \left[\frac{d+r_-}{2}\right] + 1$.

The right choices for N' and N'' are $N' = N_0' + 2(\tilde{N}_3 + \underline{N}_4)$ and $N'' = 2N_0' + N_0'' + 2(\tilde{N}_1 + \tilde{N}_3 + \tilde{N}_4 + \underline{N}_2 + \underline{N}_3 + 2\underline{N}_4)$. The Theorem 1.1 is proved.

5. The parametrix of an elliptic operator

Lemma 5.1. *Let* $f \in \mathcal{S}^m(\mathbb{R}^d)$ *and assume that the hypothesis* (1.4), (1.5) *and* (1.6) *hold with* $\overline{N} = \overline{N}_0 := 2d + 2 + 2(N_3 + N_4)$, $\overline{N}' = \overline{N}_0' := 2d + 2 + 2N_4 + 1$ *and* $\overline{N}'' = \overline{N}_0'' := 3d + 4 + 2N_2$, *where* $N_1 = N_2 = \left[\frac{d}{2}\right] + 1, N_3 = N_4 = \left[\frac{d+|m|}{2}\right]$. *Then there exists* $g_0 \in \mathcal{S}^{-m}(\mathbb{R}^d)$ *such that* $r := f\#^B g_0 - 1 \in \mathcal{S}^{-1}(\mathbb{R}^d)$, $r' := g_0\#^B f - 1 \in \mathcal{S}^{-1}(\mathbb{R}^d)$ *and* $\mathcal{O}_p^A(r)$ *and* $\mathcal{O}_p^A(r')$ *are compact operators on* $L^2(\mathbb{R}^d)$.

Proof. Set $g_0 := \frac{1}{f}$; then $g_0 \in \mathcal{S}^{-m}(\mathbb{R}^d)$ and it satisfies the condition (1.6) with the same $\overline{N}'_0, \overline{N}''_0$. Proposition 3.1 leads us to

$$\lim_{|x|\to\infty} \partial_x^\gamma \partial_\xi^\delta r(x,\xi) = 0,\ \forall \xi \in \mathbb{R}^d, \forall \gamma, \delta \in \mathbb{N}^d, |\gamma| \leq 2d+2, |\delta| \leq 3d+4.$$

We conclude by invoking Corrolary 2.1. □

Lemma 5.2. *Let $p, q \in \mathbb{N}$ and $f \in \mathcal{S}^m(\mathbb{R}^d)$. Assume that the hypothesis* (1.4), (1.5) *and* (1.6) *hold with $\overline{N} = \underline{N}_0 := 3d+7+2[|m|]+p$, $\overline{N}' = \underline{N}'_0 := 2d+6+2[|m|]+p$ and $\overline{N}'' = \underline{N}''_0 := 4d+10+p+q$. Then there exist $g', g'' \in \mathcal{S}^{-m}(\mathbb{R}^d)$ and $\overline{r}', \overline{r}'' \in \mathcal{S}^{-\infty}(\mathbb{R}^d)$ such that*

$$\overline{r}' = f\#^B g' - 1,\ \overline{r}'' = g''\#^B f - 1,$$

$$\lim_{|x|\to\infty} \partial_x^\gamma \partial_\xi^\delta \overline{r}'(x,\xi) = 0,\ \text{for all } \xi \in \mathbb{R}^d, \gamma, \delta \in \mathbb{N}^d, |\gamma| \leq p, |\delta| \leq q \qquad (5.1)$$

and

$$\lim_{|x|\to\infty} \partial_x^\gamma \partial_\xi^\delta \overline{r}''(x,\xi) = 0,\ \text{for all } \xi \in \mathbb{R}^d, \gamma, \delta \in \mathbb{N}^d, |\gamma| \leq p, |\delta| \leq q. \qquad (5.2)$$

Proof. We use the notations from the previous lemma: $g_0 := \frac{1}{f} \in \mathcal{S}^{-m}(\mathbb{R}^d)$ and $r := f\#^B g_0 - 1 \in \mathcal{S}^{-1}(\mathbb{R}^d)$. Set $r_k = r\#^B r\#^B \ldots \#^B r$ (k factors). Then $r_k \in \mathcal{S}^{-k}(\mathbb{R}^d)$, $r_{k+1} = r_k\#^B r$ and it is easy to prove by induction, using Proposition 3.1 (for r) and Proposition 3.2 (for $r_k\#^B r$) that

$$\lim_{|x|\to\infty} \partial_x^\gamma \partial_\xi^\delta r_k(x,\xi) = 0,\ \forall \xi \in \mathbb{R}^d, \forall k \geq 1, \forall \gamma, \delta \in \mathbb{N}^d, |\gamma| \leq p, |\delta| \leq q. \quad (5.3)$$

Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ with $\psi(\xi) = 1$ for $|\xi| \leq 1$ and consider a sequence $(t_k)_{k\geq 1}, t_k \in \mathbb{R}, t_k \nearrow \infty$ as $k \to \infty$ so rapidly such that the series $\sum_{k=1}^\infty s_k$ converge in $\mathcal{S}^{-1}(\mathbb{R}^d)$, where $s_k(x,\xi) := (-1)^k (1-\psi)\left(\frac{\xi}{t_k}\right) r_k(x,\xi)$. The sum s of this series satisfies

$$\lim_{|x|\to\infty} \partial_x^\gamma \partial_\xi^\delta s(x,\xi) = 0,\ \forall \xi \in \mathbb{R}^d, \forall \gamma, \delta \in \mathbb{N}^d, |\gamma| \leq p, |\delta| \leq q. \qquad (5.4)$$

If we choose $g' := g_0\#^B(1+s)$ and define $\overline{r}' := f\#^B g' - 1$, we get the equation (5.1) fulfilled. The statements regarding g'' and $\overline{r}''$ are proved in a similar way. □

Proof of Theorem 1.2. In the setting of Lemma 5.2, let us denote $\Delta := g' - g'' \in \mathcal{S}^{-m}(\mathbb{R}^d)$. We also have

$$\Delta = g''\#^B \overline{r}' - \overline{r}''\#^B g' \in \mathcal{S}^{-\infty}(\mathbb{R}^d), \qquad (5.5)$$

and via Proposition 3.2

$$\lim_{|x|\to\infty} \partial_x^\alpha \partial_\xi^\beta \Delta(x,\xi) = 0,\ \forall \xi \in \mathbb{R}^d, \forall \alpha,\beta \in \mathbb{N}^d, |\alpha| \le M', |\beta| \le M'', \tag{5.6}$$

where $M' = p - 2 - 2\left[\frac{d+m_-}{2}\right]$, $M'' = q - p - 4 - 4\left[\frac{d}{2}\right]$. Notice that

$$g' \#^B f = 1 + \overline{r}'' + \Delta \#^B f. \tag{5.7}$$

We have $\Delta \#^B f \in \mathcal{S}^{-\infty}(\mathbb{R}^d)$ and by Proposition 3.2,

$$\lim_{|x|\to\infty} \partial_x^\alpha \partial_\xi^\beta (\Delta \#^B f)(x,\xi) = 0,\ \forall \xi \in \mathbb{R}^d, \forall \alpha,\beta \in \mathbb{N}^d, |\alpha| \le \overline{M}', |\beta| \le \overline{M}'', \tag{5.8}$$

where $\overline{M}' = p - 4 - 2\left(\left[\frac{d+m_+}{2}\right] + \left[\frac{d+m_-}{2}\right]\right)$, $\overline{M}'' = q - 2p - 6 - 8\left[\frac{d}{2}\right] + 2\left[\frac{d+m_-}{2}\right]$.

Set $g := g', a := \overline{r}', b := \overline{r}'' + \Delta \#^B f$ and choose $\overline{N} = \underline{N}_0, \overline{N}' = \underline{N}_0', \overline{N}'' = \underline{N}_0''$ (with $\underline{N}_0, \underline{N}_0', \underline{N}_0''$ defined in Lemma 5.2). The numbers p, q are chosen large enough such that $\overline{M}' \ge N'$, $\overline{M}'' \ge N''$ (with N' and N'' from Theorem 1.1 where $m = -\infty$). The proof is finished.

□

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