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Unicity properties and algebraic properties for the solutions of the functional equation $f \circ f + af + b1_{\mathbb{R}} = 0$ (I)

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To Professor Ion Colojoară on the occasion of his 80th birthday

Abstract - Throughout this paper we shall deal with the functional equation in the title, for real a, b and $b \neq 0$. This functional equation was completely solved in a previous paper. Namely, we found all continuous solutions of the aforementioned functional equation. In this paper we give conditions of uniqueness for the solutions.

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1. Introduction

In this part results (without proof) from [2] which will be used throughout this paper are introduced.
Let a, b be real numbers $a \neq 0$, $b \neq 0$. We shall be concerned with the functional equation (called *fundamental equation*)

$$f \circ f + af + b1_{\mathbb{R}} = 0.$$

Namely, we want to find a continuous function $f : \mathbb{R} \to \mathbb{R}$ having the property that, for any $x \in \mathbb{R}$

$$f\left(f(x)\right) + af(x) + bx = 0.$$

Such a function (in case it exists) will be called *a solution of the fundamental equation* (or, simply *a solution*). In the sequel, the fundamental equation will be written in the form

$$f \circ f + af + bx = 0.$$

It is seen that a solution must be a homeomorphism. Moreover, the function $g = f^{-1}$ satisfies the equation

$$g \circ g + \frac{a}{b}\, g + \frac{1}{b}\, 1_{\mathbb{R}} = 0.$$

Incidentally, the fundamental equation will be written alternatively

$$f \circ f \pm af \pm bx = 0.$$

with positive a and b. The *characteristic equation* of the problem is the quadratic equation

$$x^2 + ax + b = 0.$$

with (complex) roots r_1, r_2 and discriminant $\Delta = a^2 - 4b$. Actually, in this paper we shall study the case when r_1, r_2 are real.

Theorem 1.1. (Calibration Theorem) *Let us assume that f is a solution and $|r_1| \leq |r_2|$. Then, for any real numbers x, y one has*

$$|r_1| \cdot |x - y| \leq |f(x) - f(y)| \leq |r_2||x - y|.$$

Lemma 1.1. *Let us assume $1 < r_1 < r_2$*

For any solution f we have the following properties:

a) $f(0) = 0$,

b) For any $x_0 \in \mathbb{R}$ one has:

$$\begin{aligned} r_1 x_0 \leq f(x_0) \leq r_2 x_0, &\quad \text{if} \quad x_0 \geq 0 \\ r_2 x_0 \leq f(x_0) \leq r_1 x_0, &\quad \text{if} \quad x_0 < 0. \end{aligned}$$

Lemma 1.2. *Let us assume that $r_2 < r_1 < 0$. For any solution f, we have the following properties:*

a) $f(0) = 0$,

b) For any $x_0 \in \mathbb{R}$ one has:

$$\begin{aligned} r_2 x_0 \leq f(x_0) \leq r_1 x_0, &\quad \text{if} \quad x_0 \geq 0 \\ r_1 x_0 \leq f(x_0) \leq r_2 x_0, &\quad \text{if} \quad x_0 < 0. \end{aligned}$$

Lemma 1.3. *Let us assume that $r_2 < r_1 < -1$. Let $0 \neq x_0 \in \mathbb{R}$ and $x_1 \in [r_2 x_0, r_1 x_0]$ (in case $x_0 > 0$) or $x_1 \in [r_1 x_0, r_2 x_0]$ (in case $x_0 < 0$). Using the coefficients of the fundamental equation we define the sequences $(x_n)_{n \geq 0}$ and $(x_{-n})_{n \geq 0}$ as follows:*

a) $x_{n+2} = -a x_{n+1} - b x_n$ with starting terms x_0 and x_1. Such a sequence is the sequence given via $x_{n+1} = f(x_n)$, with starting term x_0 (see Lemma 1.2 where we take $x_1 = f(x_0)$).

b) The sequence$(x_{-n})_n$ is defined in two steps:

Firstly we define the sequence $(y_n)_{n\geq 0}$ *given via*

$$y_{n+2} = -\frac{a}{b}y_{n+1} - \frac{1}{b}y_n$$

with starting terms $y_0 = x_1$ *and* $y_1 = x_0$.

Next we write $x_{-n} = y_{n+1}$ *for all natural* n. *Hence*

$$x_{-n-2} = -\frac{a}{b}x_{-n-1} - \frac{1}{b}x_{-n}$$

with starting terms $x_0 = y_1$ *and* $x_{-1} = y_2$.
Such a sequence is the sequence given via $x_{-n-1} = f^{-1}(x_{-n})$ *with starting term* x_0 *(see Lemma 1.2 where we take* $x_1 = f(x_0) \Leftrightarrow x_0 = f^{-1}(x_1)$*).*

In case $x_0 > 0$ *we have* $x_{2n} \uparrow \infty$ *(strictly),* $x_{2n+1} \downarrow -\infty$ *(strictly),* $x_{-2n} \downarrow 0$ *(strictly) and* $x_{-2n+1} \uparrow 0$ *(strictly). This implies*

$$\bigcup_{n\geq 0}([x_{2n}, x_{2n+2}] \cup [x_{-2n}, x_{-2n+2}]) = (0, \infty).$$

$$\bigcup_{n>0}([x_{2n+1}, x_{2n-1}] \cup [x_{-2n+1}, x_{-2n-1}]) = (-\infty, 0).$$

The case $x_0 < 0$ *is symmetric (e. g.* $x_{2n} \downarrow -\infty$ *strictly a.s.o.).*

Lemma 1.4. *Let us assume that* $1 < r_1 < r_2$. *Let* $0 \neq x_0 \in \mathbb{R}$ *and* $x_1 \in [r_1x_0, r_2x_0]$ *(in case* $x_0 > 0$*) or* $x_1 \in [r_2x_0, r_1x_0]$ *(in case* $x_0 < 0$*).*

We define the sequences $(x_n)_{n\geq 0}$ *respectively* $(x_{-n})_{n\geq 0}$ *exactly like in Lemma 1.3. In particular we can take* $x_{n+1} = f(x_n)$ *with starting term* x_0 *and* $x_{-n-1} = f^{-1}(x_{-n})$ *with starting term* x_0 *(see Lemma 1.1).*

In case $x_0 > 0$ *we have* $x_n \uparrow \infty$ *(strictly),* $x_{-n} \downarrow 0$ *(strictly). In case* $x_0 < 0$ *we have* $x_n \downarrow -\infty$ *(strictly) and* $x_{-n} \uparrow 0$ *(strictly). This implies*

$$\bigcup_{n\in\mathbb{Z}}[x_n, x_{n+1}] = (0, \infty), \quad \text{if} \quad x_0 > 0,$$

$$\bigcup_{n\in\mathbb{Z}}[x_n, x_{n+1}] = (-\infty, 0), \quad \text{if} \quad x_0 < 0.$$

Lemma 1.5. *Let us assume that* $0 < r_1 < 1 < r_2$. *Let* $x_0 \in \mathbb{R}$ *and* $x_1 > r_1x_0$, $x_1 > r_2x_0$. *We define the sequence* $(x_n)_{n\geq 0}$ *and* $(x_{-n})_{n\geq 0}$ *exactly like in Lemma 1.3. In particular we can take* $x_{n+1} = f(x_n)$, *with starting term* x_0 *and* $x_{-n-1} = f^{-1}(x_{-n})$, *with starting term* x_0.

Then $x_n \uparrow \infty$ *(strictly),* $x_{-n} \downarrow -\infty$ *(strictly) and*

$$\bigcup_{n\in\mathbb{Z}}[x_n, x_{n+1}] = \mathbb{R}.$$

Theorem 1.2. *(Case $1 < r_1 < r_2$).*

We shall write the fundamental equation in the form:

$$f \circ f - af + bx = 0.$$

All the solutions $f : \mathbb{R} \to \mathbb{R}$ are of the form

$$f(x) = \begin{cases} F_1(x), & \text{if } \ x > 0 \\ 0, & \text{if } \ x = 0 \\ F_2(x), & \text{if } \ x < 0, \end{cases}$$

where F_1 and F_2 are constructed as follows:

1. *We construct the sequences $(x_n)_{n\geq 0}$ and $(x_{-n})_{n\geq 0}$ according to Lemma 1.4 starting with an arbitrary $x_0 > 0$ and $x_1 \in [r_1 x_0, r_2 x_0]$. We consider a bijection $f_0 : [x_0, x_1] \to [x_1, x_2]$ having the property that for any $x > y$ in $[x_0, x_1]$ one has*

$$r_1(x - y) \leq f_0(x) - f_0(y) \leq r_2(x - y). \tag{1.1}$$

Then, for any natural n, one can construct the bijections

$$f_n : [x_n, x_{n+1}] \to [x_{n+1}, x_{n+2}]$$

and

$$f_{-n} : [x_{-n}, x_{-n+1}] \to [x_{-n+1}, x_{-n+2}]$$

defined via

$$f_{n+1}(x) = ax - bf_n^{-1}(x) \qquad \text{and} \qquad f_{-n-1}^{-1}(x) = \frac{a}{b}x - \frac{1}{b}f_{-n}^{-1}(x). \tag{1.2}$$

Finally, for any $x \in (0, \infty) = \bigcup_{n\in\mathbb{Z}} [x_n, x_{n+1}]$ we have, for some natural n:

– either $x \in [x_n, x_{n+1}]$ and $F_1(x) = f_n(x)$

– or $x \in [x_{-n}, x_{-n+1}]$ and $F_1(x) = f_{-n}(x)$.

The values at the common endpoints coincide.

2. *We construct the sequences $(x_n)_{n\geq 0}$ and $(x_{-n})_{n\geq 0}$ according to Lemma 1.4 starting with an arbitrary $x_0 < 0$ and $x_1 \in [r_2 x_0, r_1 x_0]$. We consider a bijection $f_0 : [x_1, x_0] \to [x_2, x_1]$ having the property (1.1) for any $x > y$ in $[x_1, x_0]$.*

Then, for any natural n one can construct the bijections

$$f_n : [x_{n+1}, x_n] \to [x_{n+2}, x_{n+1}]$$

and

$$f_{-n} : [x_{-n+1}, x_{-n}] \to [x_{-n+2}, x_{-n+1}]$$

defined via (1.2).

Finally, for any $x \in (-\infty, 0) = \bigcup_{n\in\mathbb{Z}} [x_{n+1}, x_n]$ *we have, for some natural* n*:*

– *either* $x \in [x_{n+1}, x_n]$ *and* $F_2(x) = f_n(x)$

– *or* $x \in [x_{-n+1}, x_{-n}]$ *and* $F_2(x) = f_{-n}(x)$.

The values at the common endpoints coincide.

Theorem 1.3. *(Case* $r_2 < r_1 < -1$*). All the solutions are obtained as follows:*

We start with an arbitrary $x_0 > 0$*, and we choose* $x_1 \in [r_2x_0, r_1x_0]$*. We apply Lemma* 1.3 *and construct the sequences* $(x_n)_n$ *and* $(x_{-n})_n$*. Let* $f_0 : [x_0, x_2] \to [x_3, x_1]$ *be a strictly decreasing bijection having the property*

$$-r_1(x - y) \leq f_0(y) - f_0(x) \leq -r_2(x - y) \tag{1.3}$$

for all $x > y$ *in* $[x_0, x_2]$.

We can construct the following strictly decreasing bijections (for any natural n *):*

$$f_{2n} : [x_{2n}, x_{2n+2}] \to [x_{2n+3}, x_{2n+1}]\,,$$

$$f_{2n}(x) = -ax - bf^{-1}_{2n-1}(x) \tag{1.4}$$

$$f_{2n+1} : [x_{2n+3}, x_{2n+1}] \to [x_{2n+2}, x_{2n+4}]\,,$$

$$f_{2n+1}(x) = -ax - bf^{-1}_{2n}(x) \tag{1.5}$$

$$f_{-2n} : [x_{-2n}, x_{-2n+2}] \to [x_{-2n+3}, x_{-2n+1}]\,,$$

$$f^{-1}_{-2n}(x) = -\frac{a}{b}x - \frac{1}{b}f_{-2n+1}(x) \tag{1.6}$$

$$f_{-2n-1} : [x_{-2n+1}, x_{-2n-1}] \to [x_{-2n}, x_{-2n+2}]\,,$$

$$f^{-1}_{-2n-1}(x) = -\frac{a}{b}x - \frac{1}{b}f_{-2n}(x). \tag{1.7}$$

Since the reunion of all above mentioned intervals is equal to $\mathbb{R} \setminus \{0\}$*, we can construct* $f : \mathbb{R} \to \mathbb{R}$*, given via:*

$$f(x) = \begin{cases} 0, & \text{if} \quad x = 0 \\ f_n(x) & \text{if} \quad x \neq 0, \end{cases}$$

where $0 \neq x$ *belongs to one of the above mentioned intervals which is the domain of definition for* f_n*,* $n \in \mathbb{Z}$*. The values at the common endpoints coincide.*

Then f *is a solution and all the solutions can be obtained in this way.*

Theorem 1.4. *Assume $0 < r_1 < 1 < r_2$ and let f be a solution with the property $f(0) \neq 0$. Then either $f(x) > x$ for any $x \in \mathbb{R}$ or $f(x) < x$ for any $x \in \mathbb{R}$.*

I. Assume that $f(x) > x$ for all $x \in \mathbb{R}$. Then f can be obtained as follows:

We construct the sequences $(x_n)_n$ and $(x_{-n})_n$ according to Lemma 1.5, where we take $x_0 = 0$ and $x_1 > 0$ arbitrary (the conditions of Lemma 1.5 are fulfilled).

We consider a strictly increasing bijection $f_0 : [0, x_1] \to [x_1, x_2]$ such that

$$r_1(x - y) \leq f_0(x) - f_0(y) \leq r_2(x - y)$$

for all $x > y$ in $[0, x_1]$.

Then, for any natural n one can construct the bijections $f_n : [x_n, x_{n+1}] \to [x_{n+1}, x_{n+2}]$ and $f_{-n} : [x_{-n}, x_{-n+1}] \to [x_{-n+1}, x_{-n+2}]$ defined via

$$f_{n+1}(x) = ax - bf_n^{-1}(x) \quad \text{şi} \quad f_{-n-1}^{-1}(x) = \frac{a}{b}x - \frac{1}{b}f_{-n}(x).$$

Finally, for any $x \in \mathbb{R} = \bigcup_{n \in \mathbb{Z}} [x_n, x_{n+1}]$ we have, for some natural n

either $x \in [x_n, x_{n+1}]$ and $f(x) = f_n(x)$,

or $x \in [x_{-n}, x_{-n+1}]$ and $f(x) = f_{-n}(x)$.

II. Assume that $f(x) < x$ for any $x \in \mathbb{R}$. Then $f^{-1}(x) > x$ for any $x \in \mathbb{R}$ and f^{-1} can be constructed according to part I.

2. Sufficient conditions for the uniqueness of the solutions

We consider the functional equation

$$f \circ f + af + bx = 0, \tag{E1}$$

where the signs of a and b are taken according to the convention from the beginning of part 1.

We shall establish what conditions can guarantee the uniqueness of the continuous solution of this functional equation. More precisely if two solutions coincide on an interval I under some conditions then they coincide everywhere. We shall see what conditions must fulfill this interval in each case.

Theorem 2.1. *Let us consider the functional equation (E1) in case $\Delta > 0$. Then we have:*

a) If $1 < r_1 < r_2$ and two continuous solutions coincide on $I = [a', b']$, $I \subset (0, \infty)$ and $\dfrac{b'}{a'} \geq r_2$, then they coincide on $(0, \infty)$.

A similar result holds if $I \subset (-\infty, 0)$.

If $I = [0, a']$, $a' > 0$, the solutions coincide on $(0, \infty)$. A similar result holds if $I = [a', 0]$, $a' < 0$.

b) If $r_2 < r_1 < -1$ and two continuous solutions coincide on $I = [a', b']$, $I \subset (0, \infty)$ and $\dfrac{b'}{a'} \geq r_2^2$ (also if $I \subset (-\infty, 0)$ and $I = [a', b']$ and $\dfrac{a'}{b'} \geq r_2^2$), then they coincide on $\mathbb{R}$.

If $0 \in I$ the two solutions coincide on $\mathbb{R}$.

c) If $r_1 < 1 < r_2$, two solutions f and g having the property that $f(0) \neq 0, g(0) \neq 0$ and coincide on $[0, a']$, $a' > 0$, coincide on $\mathbb{R}$. We have a similar result for $[a', 0]$, $a' < 0$.

Proof. Because the solutions are continuous we can use the corresponding theorems of existence from the previous paragraph.

a) We consider the equation $f \circ f - af + bx = 0$ with solutions given by Theorem 1.2.

Let f and g be two solutions that coincide on $[a', b']$, $[a', b'] \subset (0, \infty)$. We shall prove that there exist $x_0, x_1 \in [a', b']$ such that the sequence $(x_n)_{n \in \mathbb{Z}}$ is that one of Theorem 1.2.

Then we choose $x_0 = a'$ and prove that $[x_0 r_1, x_0 r_2] \subset [a', b']$. Indeed $x_0 r_1 > a' \Leftrightarrow x_0 r_1 > x_0 \Leftrightarrow r_1 > 1$ and $x_0 r_2 < b' \Leftrightarrow b' > a' r_2$. These conditions are fulfilled from the hypothesis. Then, because:

$$r_1(x - y) \leq f(x) - f(y) \leq r_2(x - y) \tag{α}$$

(the same for g), we can take $f_0 = f|_{[x_0, x_1]} = g|_{[x_0, x_1]} = g_0$.

We define $(x_n)_{n \in \mathbb{Z}}$ like in Theorem 1.2, $f_0 : [x_0, x_1] \to [x_1, x_2]$ is increasing, bijective and satisfies (α). The same for g_0, hence f_0 and g_0 fulfill the condition from Theorem 1.2.

We shall prove inductively that $f_n(x) = g_n(x)$ for $x \in [x_n, x_{n+1}]$, $n \geq 0$, where f_n and g_n are those of Theorem 1.2. We shall prove that the functions obtained in this way are increasing, bijective, continuous and satisfy (α) for all $n \geq 0$.

Let us suppose that $f_{n-1} = g_{n-1}$. We have $f_n(x) = ax - b f_{n-1}^{-1}(x)$ and $g_n(x) = ax - b f_{n-1}^{-1}(x)$. Because $f_{n-1}^{-1} = g_{n-1}^{-1}$ we have $f_n(x) = g_n(x)$, for all $x \in [x_n, x_{n+1}]$. Then $f = g$ on $[x_n, x_{n+1}]$, i. e. $f = g$ on $[x_0, \infty)$. Let us prove now that $f_{-n} = g_{-n}$, for all $n \geq 0$. Assume inductively $f_{-n+1} = g_{-n+1}$ (f_{-n} and g_{-n} are those of Theorem 1.2). But $f_{-n}^{-1}(x) = \dfrac{1}{b}\left(ax - f_{-n+1}(x)\right)$ for $x \in [x_{-n+1}, x_{-n+2}]$ and $g_{-n}^{-1}(x) = \dfrac{1}{b}\left(ax - g_{-n+1}(x)\right)$ for $x \in [x_{-n+1}, x_{-n+2}]$.

Because $f_{-n+1} = g_{-n+1}$ we have $f_{-n}^{-1} = g_{-n}^{-1}$, hence $f_{-n}(x) = g_{-n}(x)$, for all $x \in [x_{-n}, x_{-n+1}]$. Then $f_{-n} = g_{-n}$, for all $n \geq 0$. This means $f = g$ on $(0, x_0)$. So $f = g$ on $(0, \infty)$.

Similarly if $[a', b'] \subset (-\infty, 0)$ it follows that $f = g$ pe $(-\infty, 0)$. Suppose now $I = [0, a']$. We take $x_1 = a'$ and $x_0 = \dfrac{a'}{r_2}$ and obviously, it follows that $x_0, x_1 \in I$.

The proof is similar to the previous one. The same proof for
$I = [a', 0]$. Then $f = g$ on $(-\infty, 0)$.

b) If $r_2 < r_1 < -1$, let us consider the equation

$$f \circ f + af + bx = 0$$

with continuous and decreasing solution which fulfills the condition

$$-r_1(x - y) \leq f(y) - f(x) \leq -r_2(x - y) \tag{β}$$

(for $x > y$). The solutions are given by Theorem 1.3.

Let f, g two solutions that coincide on $[a', b'] \subset (0, \infty)$. Let us prove that there exist $x_1, x_2 \in [a', b']$ such that $r_2^2 x_0 \leq x_2 \leq r_1^2 x_0$ and $x_1 \in [r_2 x_0, r_1 x_0]$ such that $x_2 + ax_1 + bx_0 = 0$.

We choose $x_0 = a'$. We have $b' \geq a' r_2^2 \Rightarrow b' \geq x_0 r_2^2$.
Because $\left[x_0 r_1^2, x_0 r_2^2\right] \subset [a', b']$, we can choose $x_2 \in \left[x_0 r_1^2, x_0 r_2^2\right]$, consequently $x_2 \in [a', b']$.

$$x_1 = -\frac{bx_0 - x_2}{a} \leq \frac{-r_1 r_2 x_0 - x_0 r_1^2}{-(r_1 + r_2)} = \frac{-x_0 r_1 (r_1 + r_2)}{-(r_1 + r_2)} = x_0 r_1.$$

Similar proof for $x_1 \geq x_0 r_2$, hence the condition $x_1 \in [x_0 r_2, x_0 r_1]$ is fulfilled.

Then we can define $(x_n)_{n \in \mathbb{Z}}$ like in Theorem 1.3.

We can consider $f_0|_{[x_0, x_2]} = g_0|_{[x_0, x_2]} = g_0$ and
$f_0, g_0 : [x_0, x_2] \to [x_3, x_1]$ fulfill the conditions from Theorem 1.3. Then we can define $(f_n)_{n \in \mathbb{Z}}$ and $(g_n)_{n \in \mathbb{Z}}$ like in Theorem 1.3. They are decreasing, continuous, bijective and satisfy (β).

The functions $f_1 : [x_3, x_1] \to [x_2, x_4]$ and $g_1 : [x_3, x_1] \to [x_2, x_4]$ are given by the formulas:

$$f_1(x) = -ax - bf_0^{-1}(x), \qquad g_1(x) = -ax - bg_0^{-1}(x).$$

Hence $f_1 = g_1$ on $[x_3, x_1]$.

Now suppose inductively that $f_{2n-1} = g_{2n-1}$ on $[x_{2n+1}, x_{2n-1}]$ and we shall prove that $f_{2n} = g_{2n}$ on $[x_{2n}, x_{2n+2}]$, where

$$f_{2n-1}, g_{2n-1} : [x_{2n+1}, x_{2n-1}] \to [x_{2n}, x_{2n+2}]$$

and

$$f_{2n}, g_{2n} : [x_{2n}, x_{2n+2}] \to [x_{2n+3}, x_{2n+1}] .$$

We know that

$$f_{2n}(x) = -ax - b \cdot f_{2n-1}^{-1}(x), \qquad g_{2n}(x) = -ax - b g_{2n-1}^{-1}(x).$$

Then it follows that $f_{2n} = g_{2n}$ on $[x_{2n}, x_{2n+2}]$.

In the same way it can be shown that $f_{2n+1} = g_{2n+1}$, where

$$f_{2n+1}, g_{2n+1} : [x_{2n+3}, x_{2n+1}] \to [x_{2n+2}, x_{2n+4}]$$

and $f_{2n+2} = g_{2n+2}$, where

$$f_{2n+2}, g_{2n+2} : [x_{2n+2}, x_{2n+4}] \to [x_{2n+5}, x_{2n+3}] .$$

Then $f_n = g_n$ for all $n \geq 0$, i. e. $f = g$ on $(-\infty, x_1] \cup [x_0, \infty)$.

We prove now that $f_{-n} = g_{-n}$ $(n \geq 0)$. First, we have:

$$f_{-1}^{-1}(x) = \frac{1}{b} \left(-ax - f_0(x)\right),$$

$$g_{-1}^{-1}(x) = \frac{1}{b} \left(-ax - g_0(x)\right),$$

where $f_{-1}, g_{-1} : [x_1, x_{-1}] \to [x_0, x_2]$.

Because $f_0 = g_0$ on $[x_0, x_2]$ it follows that$f_{-1}^{-1} = g_{-1}^{-1}$ on $[x_0, x_2]$, so $f_{-1} = g_{-1}$ pe $[x_1, x_{-1}]$. Now, suppose inductively that $f_{-2n+1} = g_{-2n+1}$ on $[x_{-2n+3}, x_{-2n+1}]$ where

$$f_{-2n+1}, g_{-2n+1} : [x_{-2n+3}, x_{-2n+1}] \to [x_{-2n+2}, x_{-2n+4}]$$

and we shall prove that $f_{-2n} = g_{-2n}$ on $[x_{-2n}, x_{-2n+2}]$. Indeed

$$f_{-2n}^{-1}(x) = \frac{1}{b} \left(-ax - f_{-2n+1}(x)\right), \qquad g_{-2n}^{-1}(x) = \frac{1}{b} \left(-ax - g_{-2n+1}(x)\right),$$

where

$$f_{-2n} : [x_{-2n}, x_{-2n+2}] \to [x_{-2n+3}, x_{-2n+1}]$$

and

$$g_{-2n} : [x_{-2n}, x_{-2n+2}] \to [x_{-2n+3}, x_{-2n+1}] .$$

Because $f_{-2n+1} = g_{-2n+1}$ on $[x_{-2n+3}, x_{-2n+1}]$ it follows that $f_{-2n}^{-1} = g_{-2n}^{-1}$ on $[x_{-2n+3}, x_{-2n+1}]$.

Then $f_{-2n} = g_{-2n}$ on $[x_{-2n}, x_{-2n+2}]$.

Similarly we can prove that $f_{-2n-1} = g_{-2n-1}$, where

$$f_{-2n-1}, g_{-2n-1} : [x_{-2n+1}, x_{-2n-1}] \to [x_{-2n}, x_{-2n+2}]$$

and $f_{-2n-2} = g_{-2n-2}$, where

$$f_{-2n-2}, g_{-2n-2} : [x_{-2n-2}, x_{-2n}] \to [x_{-2n+1}, x_{-2n-1}] .$$

Thus it follows that $f_{-n} = g_{-n}$, for all $n \geq 0$.

Then it follows that $f = g$ on $(x_1, 0) \cup (0, x_0)$.

Obviously $f(0) = g(0) = 0$ and so $f = g$.

Let f and g two solutions which coincide on $[a', b'] \subset (-\infty, 0)$. We shall prove that there exist $x_1, x_3 \subset [a', b']$ with $x_3 \in \left[x_1 r_2^2, x_1 r_1^2\right]$ and $x_2 \in [x_1 r_1, x_1 r_2]$, respectively $x_0 \in \left[\frac{x_1}{r_2}, \frac{x_1}{r_1}\right]$ such that $x_3 + a x_2 + b x_1 = 0$ and $x_2 + a x_1 + b x_0 = 0$.

We choose $x_1 = b'$. Because $\left[b' r_2^2, b' r_1^2\right] \subset [a', b']$, we can choose $x_3 \in \left[b' r_2^2, b' r_1^2\right]$ and then $x_3 \in [a', b']$.

We choose $x_2 = -\dfrac{x_3 - b x_1}{a}$. We must prove that $x_2 \in [x_1 r_1, x_1 r_2]$.

But $x_2 \leq \dfrac{-x_1 r_2^2 - r_1 r_2 x_1}{-(r_1 + r_2)} = \dfrac{-x_1 r_2 (r_1 + r_2)}{-(r_1 + r_2)} = x_1 r_2$.

Similarly we have: $x_2 \geq x_1 r_1$.

We choose

$$x_0 = \frac{-a x_1 - x_2}{b} \leq \frac{(r_1 + r_2) x_1 - x_1 r_1}{r_1 r_2} = \frac{x_1}{r_1}.$$

Similarly we have: $x_0 \geq \dfrac{x_1}{r_2}$, hence $x_1 \in [x_0 r_2, x_0 r_1]$.

So we can define $(x_n)_{n \in \mathbb{Z}}$ like in Theorem 1.3. Let us prove that $f = g$ on $[x_0, x_2]$.

We denote $f_1 = f|_{[x_3, x_1]}$ and $g_1 = g|_{[x_3, x_1]}$. Obviously $f_1 = g_1$.

The functions $f_1, g_1 : [x_3, x_1] \to [x_2, x_4]$ are continuous, bijective and fulfill the relationship (β).

We define the function $h : [x_3, x_1] \to \mathbb{R}$ (we shall see that one can consider $h : [x_3, x_1] \to [x_0, x_2]$)

$$h(x) = -\frac{f_1(x) + a x}{b} = -\frac{g_1(x) + a x}{b}.$$

The function h is continuous; because f_1 and g_1 fulfill (β), it follows that h fulfills the relationship:

$$\frac{x - y}{r_1} \leq h(x) - h(y) \leq \frac{x - y}{r_2} \qquad (x > y).$$

Hence h is decreasing on $[x_3, x_1]$ and

$$h(x_3) = -\frac{f_1(x_3) + a x_3}{b} = -\frac{x_4 + a x_3}{b} = x_2;$$

$$h(x_1) = -\frac{f_1(x_1) + ax_1}{b} = -\frac{x_2 + ax_1}{b} = x_0.$$

Therefore $h : [x_3, x_1] \to [x_0, x_2]$ is bijective.

We denote $f_0 = h^{-1}$ and $g_0 = h^{-1}$; $f_0, g_0 : [x_0, x_2] \to [x_3, x_1]$. It follows that $f_0 = g_0$ and obviously

$$f_1(x) = -ax - bf_0^{-1}(x), \qquad g_1(x) = -ax - bg_0^{-1}(x).$$

It is clear that $x_0 r_1^2 \leq x_2 \leq x_0 r_2^2$.

So f and g coincide on $[x_0, x_2]$, where $(x_n)_{n\in\mathbb{Z}}$ is defined like in Theorem 1.3.

Using the same reasoning as in the previous case it will follow that f and g coincide on $\mathbb{R}$.

c) If $r_1 < 1 < r_2$, we consider the equation

$$f \circ f - af + bx = 0$$

with continuous and increasing solutions given by Theorem 1.4 (case $f(x) > x$).

Let f and g two solutions which coincide on $[0, a']$. It follows from the hypothesis that f and g have no fixed points.

If $f(x) > x$, for any $x \in \mathbb{R}$ it follows that $g(x) > x$, for any $x \in \mathbb{R}$ (f and g coincide on $[0, a']$).

We choose $x_1 = a'$, $x_0 = 0$. We can define $(x_n)_{n\in\mathbb{Z}}$ like in Theorem 1.4; $x_n \underset{n}{\to} \infty$ and $x_{-n} \underset{n}{\to} -\infty$.

We can define f_0 and g_0 like this: $f_0 = f|_{[0,x_1]}$ and $g_0 = g|_{[0,x_1]}$. The functions $f_0, g_0 : [0, x_1] \to [x_1, x_2]$ are continuous, bijective and fulfill (α).

Defining $(f_n)_{n\in\mathbb{Z}}$ like in Theorem 1.4, it will follow that f_n are continuous, bijective and fulfill (α).

Similarly to a) one can prove inductively that $f_n = g_n$, for all $n \in \mathbb{Z}$. Therefore $f = g$ on $\mathbb{R}$.

We shall prove now that if f and g coincide on $[b', 0]$, they coincide on $\mathbb{R}$ (where $b' < 0$).

Let $(x_n)_{n\in\mathbb{Z}}$ the sequence which appears in the construction of the solution in Theorem 1.4. We choose $x_0 = 0$ and $x_1 = -b \cdot b'$, $x_1 > 0$.
Let us prove that $f(x) = g(x)$ for $x \in [0, -b \cdot b']$.
We have: $x_1 - ax_0 + bx_{-1} = 0$; $x_0 = 0 \Rightarrow x_{-1} = b'$. Hence, considering $(f_n)_{n\in\mathbb{Z}}$ and $(g_n)_{n\in\mathbb{Z}}$ the functions which appear in the construction of the solution in Theorem 1.4 ; $f_{-1} : [x_{-1}, 0] \to [0, x_1]$ and $g_{-1} : [x_{-1}, 0] \to [0, x_1]$ it is clear that $f_{-1} = g_{-1}$ implies the fact that $f_{-1}^{-1}(x) = g_{-1}^{-1}(x)$ for $x \in [0, x_1]$.

But

$$f_{-1}^{-1}(x) = -\frac{1}{b}\,(ax + f_0(x)),\; g_{-1}^{-1}(x) = -\frac{1}{b}\,(ax + g_0(x)),$$

where $f_0 : [0, x_1] \to [x_1, x_2]$; $g_0 : [0, x_1] \to [x_1, x_2]$. Hence $f_0(x) = g_0(x)$, for all $x \in [0, x_1]$. According to the fact which has been proved, it follows that $f = g$ on $\mathbb{R}$.

The case $f(x) < x$, for all $x \in \mathbb{R}$, is similar. □

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