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The three equivalent forms of completely positive maps on matrices

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Dedicated to Professor Ion Colojoară, mentor and friend,
on the occasion of his 80th birthday

Abstract - Motived by the importance of quantum operations in quantum information theory, we rigorously present the three equivalent (Stinespring, Kraus, and Choi) forms of completely positive maps on full C^*-algebras of matrices, as well as their connection with the Arveson's Radon-Nikodym derivative. In order to make this accessible to a broader audience we employ mostly linear algebra facts and carefully review the prerequisites.

Key words and phrases : C^*-algebra, completely positive map, tensor product, Stinespring representation, Kraus form, Choi's matrix, Radon-Nikodym derivative, quantum operation

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1. Introduction

In modern quantum physics, the formalism of quantum operations can be used to describe a very large class of dynamical evolution of quantum systems, e.g. see K. Kraus [7], E.B. Davies [4]. Also, there is a recent interest in quantum information theory in connection to quantum operations that can be used to model quantum channels, quantum measurements, and many others, see D. Leung [8] and the bibliography cited there. In quantum information theory a quantum operation is a linear map $\varphi\colon M_n \to M_k$ (here M_k denotes the C^*-algebra of all $k \times k$ complex matrices) that is trace non-increasing and completely positive. The requirement of complete positivity is justified by the fact that if a state is entangled by another state, mathematically expressed as a tensor product, the output state should be a valid state as well.

The three equivalent forms of a completely positive map φ on matrices are the following:

(S) (Stinespring) $\varphi(A) = V^*\pi(A)V$, where π is a $*$-representation and V a matrix of appropriate size.

(K) (Kraus) $\varphi(A) = \sum_j V_j^* A V_j$, where V_j are matrices of the same appropriate size.

(C) (Choi) $\varphi(A) = \sum_{r,s} \varphi_{r,s} \mathcal{E}_r^* A \mathcal{E}_s$, where $\Phi = [\varphi_{r,s}]$ is a positive matrix and $\mathcal{E}_r$ are matrix units of appropriate sizes.

The aim of this article is to rigorously show the equivalence of these three forms for completely positive maps on matrices. Our approach uses the technique of the Arveson's Radon-Nikodym derivative following [10] and [5]. The Stinespring form, as well as the Kraus form, are more general objects but, in this finite dimensional setting, we view them in a more elementary way that make use only of linear algebra notions.

The reader may have knowledge of the more advanced monograph of V. Paulsen [9] on completely positive maps, but this is not necessary. We prefer to keep the prerequisites to a minimum by assuming that the reader has a good command on linear algebra, for example, as in the S. Axler's linear algebra textbook [2], but any other (more advanced) textbook on linear algebra is sufficient. In order to make this article useful for a broader audience, we carefully recall the prerequisites: the C^*-algebra M_k, tensor products, and the Arveson's Radon-Nikodym derivative. Since we addressed this article to mathematicians but do not want to exclude the physicists from our potential readers, we also briefly indicated the correspondence between the Dirac formalism and the mathematical formalism that we employ in this article.

2. Notation and Preliminaries

This section reviews the notation and the linear algebra prerequisites that are necessary for reading the material on completely positive maps.

2.1. The Inner Product Space $\mathbb{C}^n$

For arbitrary natural number n let $\mathbb{C}^n$ denote the vector space over the complex field $\mathbb{C}$ of complex column vectors with n entries $x = (\xi_j)_{j=1}^n$. On this vector space we consider the *inner product*

$$\langle x, y\rangle = \sum_{j=1}^n \xi_j \overline{\eta}_j, \quad x = (\xi_j)_{j=1}^n,\ y = (\eta_j)_{j=1}^n. \tag{2.1}$$

Note that, in this notation, the inner product $\langle \cdot, \cdot\rangle$ is linear in the first variable and conjugate linear in the second variable.

We denote by $\|\cdot\|$ the associated unitary *norm*, that is,

$$\|x\| = \left(\sum_{j=1}^{n} |\xi_j|^2\right)^{1/2}, \quad x = (\xi_j)_{j=1}^{n}, \tag{2.2}$$

and by $\{e_i^{(n)}\}_{i=1}^n$ the *canonical basis* of $\mathbb{C}^n$, that is, $e_i^{(n)}$ is the n-tuple with 1 on the i-th position and 0 elsewhere.

2.2. The Vector Space $M_{k,n}$

For arbitrary natural numbers k and n we denote by $M_{k,n}$ the vector space over the field $\mathbb{C}$ of $k\times n$ matrices with complex entries. We identify in a natural way $M_{k,n}$ with the vector space $\mathcal{L}(\mathbb{C}^n,\mathbb{C}^k)$ of linear transformations $A\colon \mathbb{C}^n\to\mathbb{C}^k$, by means of the canonical bases $\{e_i^{(k)}\}_{i=1}^k$ and $\{e_j^{(n)}\}_{j=1}^n$, more precisely, the identification is $A=[a_{i,j}]_{i=\overline{1,k},j=\overline{1,n}}$ where

$$a_{i,j} = \langle Ae_j^{(n)}, e_i^{(k)}\rangle, \quad i=1,\ldots,k,\ j=1,\ldots,n. \tag{2.3}$$

By this identification, on $M_{k,n}$ there exists the *operator norm*, more precisely,

$$\begin{aligned}\|A\| &= \sup\{\|Ax\| \mid x\in\mathbb{C}^n,\ \|x\|\leq 1\} \\ &= \inf\{t\geq 0 \mid \|Ax\|\leq t\|x\| \text{ for all } x\in\mathbb{C}^n\}.\end{aligned} \tag{2.4}$$

This norm makes $M_{k,n}$ a (complete) normed space.

On $M_{k,n}$ we consider the *adjoint operation*, more precisely, $M_{k,n}\ni A\mapsto A^*\in M_{n,k}$, where the matrix of A^* is obtained by changing rows into columns in the matrix of A and taking the complex conjugate. In terms of the identification of $M_{k,n}$ with the vector space $\mathcal{L}(\mathbb{C}^n,\mathbb{C}^k)$, this means

$$\langle Ax, y\rangle = \langle x, A^*y\rangle, \quad x\in\mathbb{C}^n,\ y\in\mathbb{C}^k. \tag{2.5}$$

The map $M_{k,n}\ni A\mapsto A^*\in M_{n,k}$ has the following properties:

- $(\alpha A+\beta B)^* = \overline{\alpha}A^*+\overline{\beta}B^*$, $A,B\in M_{k,n}$, $\alpha,\beta\in\mathbb{C}$;
- $(AB)^* = B^*A^*$, $A\in M_{k,n}$ and $B\in M_{n,m}$;
- $(A^*)^* = A$, $A\in M_{k,n}$.

With respect to the canonical bases of $\mathbb{C}^n$ and $\mathbb{C}^k$, for $n,k\in\mathbb{N}$, we consider the matrix units $\{E_{i,j}^{(n,k)} \mid i=1,\ldots,n,\ j=1,\ldots,k\}\subset M_{n,k}$ of size $n\times k$, that is, $E_{i,j}^{(n,k)}$ is the $n\times k$ matrix with all entries 0 except the (i,j)-th entry which is 1. In case $n=k$, we denote simply $E_{i,j}^{(n)}=E_{i,j}^{(n,n)}$.

We also record the following direct consequences of the definitions: for all $j = 1, \ldots, n$ and $i = 1, \ldots, k$ we have

$$E_{i,j}^{(n,k)^*} = E_{j,i}^{(k,n)}, \tag{2.6}$$

and if, in addition, $p \in \mathbb{N}$, $r = 1, \ldots, k$, and $s = 1, \ldots, p$, then

$$E_{i,j}^{(n,k)} E_{r,s}^{(k,p)} = \delta_{j,r} E_{i,s}^{(n,p)}. \tag{2.7}$$

2.3. The C^*-Algebra M_k

We denote $M_k = M_{k,k}$ and note that it is an algebra over the complex field. On M_k we consider the adjoint operation $*$ which now it is internal $M_k \ni A \mapsto A^* \in M_k$. Thus, M_k is a unital $*$-algebra; we denote by I_k its unit, that is, the matrix with 1 on the main diagonal and 0 elsewhere.

A matrix $A \in M_k$ is called *selfadjoint* (*hermitian*) if $A = A^*$. If A is selfadjoint then all its eigenvalues are simple and real. A matrix $A \in M_k$ is called *positive* if it is selfadjoint and all its eigenvalues are nonnegative. We denote by M_k^+ the set of positive matrices from M_k.

Proposition 2.1. *Let $A \in M_k$. The following assertions are equivalent:*

(i) A is positive.

(ii) $A = B^* B$ for some $B \in M_k$.

(iii) $A = B^2$ for some $B \in M_k^+$.

(iv) $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{C}^k$.

Given $A \in M_k^+$, the matrix $B \in M_k^+$ such that $A = B^2$, as in item (iii) of Proposition 2.1, is unique, and it is denoted by $A^{1/2}$. From the spectral point of view, A and $A^{1/2}$ have same kernel and any eigenvalue of $A^{1/2}$ is of the form $\lambda^{1/2}$ for λ an eigenvalue of A, with the same multiplicity. Clearly, M_k^+ is a *convex cone*, that is, $\alpha A + \beta B \in M_k^+$ for any $A, B \in M_k^+$ and any $\alpha, \beta \geq 0$. In addition, it is also *strict*, that is, $M_k^+ \cap (-M_k^+)$ consists only on the null matrix.

The cone M_k^+ induces an *order* on the set of all selfadjoint $k \times k$ matrices M_k^{h}. More precisely, $A \geq 0$ for all $A \in M_k^+$ and, if $B, C \in M_k^{\mathrm{h}}$ we have $B \geq C$, by definition, if $B - C \in M_k^+$. In view of Proposition 2.1.(iv), this order relation can be defined in terms of the action of M_k on $\mathbb{C}^k$. More precisely, $B \geq C$ if and only if $\langle Bx, x\rangle \geq \langle Cx, x\rangle$ for all $x \in \mathbb{C}^k$.

M_k^{h} is a vector space over the field of real numbers. In addition, the cone M_k^+ generates M_k^{h}, more precisely:

Proposition 2.2. *Any $A \in M_k^{\mathrm{h}}$ can be written as a difference of two positive matrices $A = A_+ - A_-$. If, in addition, we require that there are no common eigenvalues of A_+ and A_-, then this decomposition is unique.*

The operator norm (2.4) makes M_k a unital normed algebra, that is,

$$\|AB\| \leq \|A\|\|B\|, \quad A, B \in M_k, \quad \|I_k\| = 1. \tag{2.8}$$

With respect to the involution $*$ the norm has an important property:

$$\|A^* A\| = \|A\|^2, \quad A \in M_k. \tag{2.9}$$

In particular, the involution is isometric, that is, $\|A^*\| = \|A\|$ for all $A \in M_k$.

On M_k there is a special linear form, the *trace* $\operatorname{tr}\colon M_k \to \mathbb{C}$ defined as the sum of the entries from the main diagonal

$$\operatorname{tr}(A) = \sum_{j=1}^{k} a_{j,j}, \quad A = [a_{i,j}]_{i,j=1}^{k} \in M_k. \tag{2.10}$$

In addition to linearity, the trace has two remarkable properties:

$$\operatorname{tr}(AB) = \operatorname{tr}(BA), \quad A, B \in M_k \text{ and } \operatorname{tr}(A) \geq 0, \quad A \in M_k^+. \tag{2.11}$$

The trace is *faithful* in the sense that if $A \in M_k^+$ and $\operatorname{tr}(A) = 0$ then $A = 0$.

2.4. Abstract Tensor Products

In this subsection we recall the definition, the construction, and the basic properties of tensor products of vector spaces.

Proposition 2.3. *Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be three vector spaces over the same field $\mathbb{K}$ and let $\tau\colon \mathcal{E} \times \mathcal{F} \to \mathcal{G}$ be a bilinear map. The following assertions are equivalent:*

(a) Let r be an arbitrary natural number and $e_1, \ldots, e_r \in \mathcal{E}$, $f_1, \ldots, f_r \in \mathcal{F}$ vectors such that

$$\sum_{j=1}^{r} \tau(e_j, f_j) = 0.$$

If $e_1, \ldots, e_r$ are linearly independent then $f_1 = f_2 = \ldots = f_r = 0$ and, symmetrically, if $f_1, \ldots, f_r$ are linearly independent then $e_1 = e_2 = \ldots = e_r = 0$.

(b) For any r, s natural numbers and for any linearly independent vectors $e_1, \ldots, e_r \in \mathcal{E}$ and $f_1, \ldots, f_s \in \mathcal{F}$ the family of vectors $\{\tau(e_i, f_j) \mid i = 1, \ldots, r,\ j = 1, \ldots, s\}$ is linearly independent in $\mathcal{G}$.

Given $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$, three vector spaces over the same field $\mathbb{K}$ and a bilinear map $\tau\colon \mathcal{E} \times \mathcal{F} \to \mathcal{G}$, the couple $(\tau; \mathcal{G})$ is called *linearly independent* if any, hence both, of the conditions (a) and (b) in Proposition 2.3 hold(s).

A *tensor product* of two vector spaces $\mathcal{E}$ and $\mathcal{F}$ over the same field $\mathbb{K}$ is a pair $(\mathcal{G}; \tau)$ such that:

- $(\mathcal{G};\tau)$ is linearly independent.
- $\tau(E\times F)$ linearly spans $\mathcal{G}$.

Theorem 2.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be two arbitrary vector spaces over the same field $\mathbb{K}$. Then:*

(i) There exists a tensor product $(\mathcal{G};\tau)$ of $\mathcal{E}$ and $\mathcal{F}$.

(ii) Let $(\mathcal{G};\tau)$ be a tensor product of $\mathcal{E}$ and $\mathcal{F}$. Then, for any vector space $\mathcal{H}$ over $\mathbb{K}$ and any bilinear map $\chi\colon\mathcal{E}\times\mathcal{F}\to\mathcal{H}$ there exists a unique linear map $\tilde{\chi}\colon\mathcal{G}\to\mathcal{H}$ such that $\tilde{\chi}\circ\tau=\chi$.

(iii) For any two tensor products $(\mathcal{G}_i;\tau_i)$ of $\mathcal{E}$ and $\mathcal{F}$, $i=1,2$, there exists a unique linear isomorphism $\chi\colon\mathcal{G}_1\to\mathcal{G}_2$ such that $\chi\circ\tau_1=\tau_2$.

The property depicted at (ii) is called the *universality property* of the tensor product. According to the property (iii) the tensor product is unique to a linear isomorphism; we use the notation $\mathcal{E}\otimes\mathcal{F}$ to denote it, more precisely, letting $(\mathcal{G};\tau)$ be the notation for the tensor product of $\mathcal{E}$ and $\mathcal{F}$ as in the definition, $\mathcal{G}=\mathcal{E}\otimes\mathcal{F}$ and $\tau(e,f)=e\otimes f$ for any $e\in\mathcal{E}$ and $f\in\mathcal{F}$.

We recall briefly one of the constructions of the tensor products. On the vector space $\mathcal{X}$ of $\mathbb{K}$-valued functions defined on $\mathcal{E}\times\mathcal{F}$ and having finite supports we consider the vector subspace $\mathcal{N}$ spanned by the functions

$$\delta_{\alpha_1e_1+\alpha_2e_2,\beta_1f_1+\beta_2f_2}-\alpha_1\beta_1\delta_{e_1,f_1}-\alpha_1\beta_2\delta_{e_1,f_2}-\alpha_2\beta_1\delta_{e_2,f_1}-\alpha_2\beta_2\delta_{e_2,f_2}, \quad (2.12)$$

where $e_1,e_2\in\mathcal{E}$, $f_1,f_2\in\mathcal{F}$, and $\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbb{K}$, and we denote by $\delta_{e,f}\colon\mathcal{E}\times\mathcal{F}\to\mathbb{K}$ the delta function supported at $(e,f)\in\mathcal{E}\times\mathcal{F}$. Then, by definition, $\mathcal{E}\otimes\mathcal{F}=\mathcal{X}/\mathcal{N}$ and $e\otimes f=\pi(\delta_{e,f})$ for all $e\in\mathcal{E}$ and $f\in\mathcal{F}$, where $\pi\colon\mathcal{X}\to\mathcal{X}/\mathcal{N}$ is the canonical projection.

Remark 2.1. *Tensor Products of Function Spaces.* Let X and Y be two nonempty sets and assume that the vector space $\mathcal{E}$ consists of functions on $e\colon X\to\mathbb{K}$ and, similarly, the vector space $\mathcal{F}$ consists on functions $f\colon Y\to\mathbb{K}$. Then the tensor product $\mathcal{E}\otimes\mathcal{F}$ can be realized as a vector space of functions on $X\times Y$, as follows. For arbitrary $e\in\mathcal{E}$ and $f\in\mathcal{F}$, define $e\otimes f\colon X\times Y\to\mathbb{K}$ by

$$(e\otimes f)(x,y)=e(x)f(y),\quad x\in X,\ y\in Y. \quad (2.13)$$

Then, letting $\mathcal{E}\otimes\mathcal{F}$ denote the vector space spanned by all elementary tensors $e\otimes f$, it is easy to see that it is a tensor product of $\mathcal{E}$ and $\mathcal{F}$.

Proposition 2.4. *Let $(e_i)_{i\in\mathcal{I}}$ be a Hamel (that is, algebraic) basis of the vector space $\mathcal{E}$ and $(f_j)_{j\in\mathcal{J}}$ be a Hamel basis of the vector space $\mathcal{F}$. Then $\{e_i\otimes f_j\mid i\in\mathcal{I},\ j\in\mathcal{J}\}$ is a Hamel basis of the vector space $\mathcal{E}\otimes\mathcal{F}$. In particular,* $\dim(\mathcal{E}\otimes\mathcal{F})=\dim(\mathcal{E})\dim(\mathcal{F})$.

One way of getting the above proposition is to note that fixing a Hamel basis $(e_i)_{i\in\mathcal{I}}$ on the vector space $\mathcal{E}$ yields an identification of $\mathcal{E}$ with the vector space of all finitely supported functions $e\colon \mathcal{I}\to\mathbb{K}$, doing a similar identification of $\mathcal{F}$ with finitely supported functions on $\mathcal{J}$, and then applying Remark 2.1 in order to get a Hamel basis $\{e_i\otimes f_j \mid i\in\mathcal{I},\ j\in\mathcal{J}\}$.

We finally recall the tensor product of linear maps. Assume that $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are vector spaces over the same field $\mathbb{K}$ and let $\varphi\colon\mathcal{E}\to\mathcal{G}$ and $\psi\colon\mathcal{F}\to\mathcal{H}$ be two linear maps. We define a new linear map $\varphi\otimes\psi\colon\mathcal{E}\otimes\mathcal{F}\to\mathcal{G}\otimes\mathcal{H}$ in the following way: for each $e\in\mathcal{E}$ and $f\in\mathcal{F}$ let $(\varphi\otimes\psi)(e\otimes f)=\varphi(e)\otimes\psi(f)$ and then extend it by linearity. It can be proven that is a correct definition (in general, the representation of an element as a linear combination as elementary tensors in not unique) and that $\varphi\otimes\psi$ is a linear map.

2.5. Tensor Products of Matrices

For finite dimensional vector spaces the tensor product has more concrete representations.

Let n and k be two natural numbers. Then the tensor product space $\mathbb{C}^n\otimes\mathbb{C}^k$ can be naturally identified with $\mathbb{C}^{nk}$ as follows: if $x=(\xi_j)_{j=1}^n$ and $y=(\eta_i)_{i=1}^k$, then the elementary tensor $x\otimes y$ is identified with the vector $(\xi_j\eta_i)_{i=1,j=1}^{n,k}$. Thus, $\mathbb{C}^n\otimes\mathbb{C}^k$ can be further identified with $M_{k,n}$.

Here and in the following we use the tensor notation for rank one operators, that is, if k and n are natural numbers and $x\in\mathbb{C}^n$ and $y\in\mathbb{C}^k$ are nontrivial vectors, then the rank 1 operator $x\otimes y\in\mathcal{L}(\mathbb{C}^k,\mathbb{C}^n)=M_{n,k}$ is defined by $(x\otimes y)z=\langle z,y\rangle x$ for all $z\in\mathbb{C}^k$.

With this notation, the system of matrix units $\{E_{i,j}^{(n,k)}\mid i=1,\ldots,n,\ j=1,\ldots,k\}\subset M_{n,k}$ that makes a basis of $M_{n,k}$ have a tensor representation in terms of the canonical bases $\{e_i^{(n)}\mid i=1,\ldots,n\}$ of $\mathbb{C}^n$ and $\{e_i^{(k)}\mid j=1,\ldots,k\}$ of $\mathbb{C}^k$, that is,

$$E_{i,j}^{(n,k)}=e_i^{(n)}\otimes e_j^{(k)},\quad i=1,\ldots,n,\ j=1,\ldots,k. \tag{2.14}$$

Let m and n be natural numbers. Initially $M_m\otimes M_n$ is only a vector space. We show that it is a C^*-algebra in a natural way. We first identify $M_m\otimes M_n$ with $M_m(M_n)$, defined as the vector space of all $m\times m$ block matrices with entries in M_n, more precisely, we identify $A\otimes B$ with the matrix

$$\begin{bmatrix} a_{1,1}B & a_{1,2}B & \ldots & a_{1,m}B \\ a_{2,1}B & a_{2,2}B & \ldots & a_{2,m}B \\ \vdots & \vdots & & \vdots \\ a_{m,1}B & a_{m,2}B & \ldots & a_{m,m}B \end{bmatrix},\quad A=[a_{i,j}]_{i,j=1}^m\in M_m,\ B\in M_n. \tag{2.15}$$

Further, we identify $M_m(M_n)$ with M_{mn} through (2.15). These identifications are $*$-isomorphisms, and thus $M_m \otimes M_n$ is a C^*-algebra, $*$-isomorphic with M_{mn}.

From the operational point of view, we point out the multiplication on elementary tensors:

$$(A \otimes B)(C \otimes D) = AC \otimes BD, \quad A, C \in M_m,\ B, D \in M_n, \tag{2.16}$$

and the involution

$$(A \otimes B)^* = A^* \otimes B^*, \quad A \in M_m,\ B \in M_n. \tag{2.17}$$

In addition, the operator norm of an elementary tensor can be easily calculated

$$\|A \otimes B\| = \|A\|\|B\|, \quad A \in M_m,\ B \in M_n. \tag{2.18}$$

The definition of the matrix $A \otimes B$ for matrices A and B is in accordance with the definition of the tensor product of linear maps as well.

2.6. Dirac Formalism vs. Tensor Product Formalism

In the formula (2.14) there is a certain abuse of notation with respect to the Hilbert space formalism. To be more precise, assume that $e \in \mathbb{C}^n$ and $f \in \mathbb{C}^k$ are two vectors and we want to define a linear operator of rank one $: \mathbb{C}^k \to \mathbb{C}^n$ with range spanned by e and null space the orthogonal of f: the classical way is to use the notation $e \otimes \overline{f}$

$$(e \otimes \overline{f})h = \langle h, f\rangle e, \quad h \in \mathbb{C}^k. \tag{2.19}$$

The bar on f is motivated by the fact that the inner product is antilinear in the second variable. Apparently, this conflicts with (2.14), but taking into account that the vectors $e_j^{(k)}$ have all their components real numbers, actually there is no contradiction here.

From this point of view, the Dirac formalism makes the difference between the vectors in an inner product $\langle e||f\rangle$ by calling $\langle e|$ a "bra" and $|f\rangle$ a "ket", and the inner product being linear in the second variable and conjugate linear in the first variable, which means only a swap of left and right arguments in the inner product. In addition to this mild change, the Dirac formalism makes a difference between vectors in the Hilbert space $\mathbb{C}^n$ and linear functionals on $\mathbb{C}^n$ via the Riesz representation theorem, by identifying a linear functional $\varphi\colon \mathbb{C}^n \to \mathbb{C}$ with the vector $z_\varphi \in \mathbb{C}^k$ such that $\varphi(\cdot) = \langle \cdot, z_\varphi\rangle$. Also, in the Dirac formalism, what we defined at (2.19) by the rank one operator $e \otimes \overline{f}$ corresponds to $|f\rangle\langle e|$.

3. Completely Positive Maps on Matrices

In this section we recall the definition of completely positive maps on matrices, their equivalence with positive semidefinte maps, the Stinespring representation, and briefly review the Arveson's Radon-Nikodym.

3.1. Definition and Examples

Let k and n be natural numbers and $\varphi\colon M_n \to M_k$ a linear map. The map φ is called *positive* if it maps positive matrices into positive matrices, briefly, $\varphi(M_n^+) \subseteq M_k^+$.

Example 3.1. The transpose map $\tau\colon M_k \to M_k$ that maps each $k \times k$ matrix into its transpose is positive.

Let, in addition, m be a natural number. A linear map $\varphi\colon M_n \to M_k$ always induces a linear map $\varphi_m = I_m \otimes \varphi\colon M_m \otimes M_n \to M_m \otimes M_k$, more precisely, with the identification $M_m \otimes M_n \simeq M_m(M_n)$, the C^*-algebra of all $m \times m$ matrices with entries from M_n, and similary the identification $M_m \otimes M_k \simeq M_m(M_k)$,

$$\varphi_m([A_{i,j}]_{i,j=1}^m) = [\varphi(A_{i,j})]_{i,j=1}^m, \quad [A_{i,j}]_{i,j=1}^m \in M_m(M_n). \tag{3.1}$$

Recall that $M_m \otimes M_n$ is a C^*-algebra in a natural way and hence positive elements are unambiguously defined. The map φ is called *m-positive* if φ_m is positive. Clearly, if φ if m-positive then it is l-positive for all natural numbers $l \leq m$, in particular, it is positive. The converse implication is not true.

Example 3.2. The transpose map $\tau\colon M_2 \to M_2$ is positive but not 2-positive. To see this let

$$A = \begin{bmatrix} E_{1,1}^{(2)} & E_{1,2}^{(2)} \\ E_{2,1}^{(2)} & E_{2,2}^{(2)} \end{bmatrix} \geq 0$$

but

$$\tau_2(A) = \begin{bmatrix} E_{1,1}^{(2)} & E_{2,1}^{(2)} \\ E_{1,2}^{(2)} & E_{2,2}^{(2)} \end{bmatrix}$$

is not positive.

A linear map $\varphi\colon M_n \to M_k$ is called *completely positive* if it is m-positive for all natural numbers m. We denote by $\mathrm{CP}(M_n, M_k)$ the set of all completely positive maps from M_n to M_k. It is easy to see that $\mathrm{CP}(M_n, M_k)$ is a strict convex cone in the vector space $\mathcal{L}(M_n, M_k)$. In particular, there is the *natural order* relation on $\mathrm{CP}(M_n, M_k)$, more precisely, given $\varphi, \psi \in \mathrm{CP}(M_n, M_k)$ we have $\varphi \leq \psi$, by definition, if $\psi - \varphi \in \mathrm{CP}(M_n, M_k)$.

The following examples will be proven to be generic.

Example 3.3. **1.** *$*$-Morphisms.* Let $\pi\colon M_n \to M_k$ be a morphism of $*$-algebras, for $n \leq k$ (if $n > k$ there are not so many!). Then π is completely positive.

2. *Stinespring Representation.* Let $\pi\colon M_n \to M_m$ be a morphism of $*$-algebras, for $n \leq m$ and $V \in M_{m,k}$. Then $\varphi = V^*\pi(\cdot)V \in \mathrm{CP}(M_n, M_k)$.

3. *Kraus Representation.* Given $n \times k$ matrices $V_1, V_2, \ldots, V_m \in M_{n,k}$ define $\varphi\colon M_n \to M_k$ by

$$\varphi(A) = V_1^* A V_1 + V_2^* A V_2 + \cdots + V_m^* A V_m \text{ for all } A \in M_n. \tag{3.2}$$

Then φ is completely positive.

3.2. Positive Semidefinite Maps

A linear map $\varphi\colon M_n \to M_k$ is called *positive semidefinite* if for any real number l, matrices $A_1, \ldots, A_l \in M_n$, and any vectors $x_1, \ldots, x_l \in \mathbb{C}^k$, we have

$$\sum_{i,j=1}^{l} \langle \varphi(A_i^* A_j) x_j, x_i \rangle \geq 0. \tag{3.3}$$

Proposition 3.1 (W.F. Stinespring [11]) *A linear map $\varphi\colon M_n \to M_k$ is positive semidefinite if and only if it is completely positive.*

Proof. Assume that $\varphi \in \mathrm{CP}(M_n, M_k)$ and let l be any natural number, matrices $A_1, \ldots, A_l \in M_n$, and vectors $x_1, \ldots, x_l \in \mathbb{C}^k$, all arbitrary. Then the block $l \times l$ matrix $A = [A_i^* A_j]_{i,j=1}^{l}$ is positive in $M_l(M_n) \simeq M_l \otimes M_n$ since

$$A = \begin{bmatrix} A_1^* & 0 & \ldots & 0 \\ A_2^* & 0 & \ldots & 0 \\ \vdots & & & \\ A_l^* & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} A_1 & A_2 & \ldots & A_l \\ 0 & 0 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & 0 \end{bmatrix}. \tag{3.4}$$

Also, letting $\mathbf{x}$ be the column vector with "entries" $x_1, \ldots, x_l$ we have

$$\sum_{i,j=1}^{l} \langle \varphi(A_i^* A_j) x_j, x_i \rangle = \langle \varphi_l(A)\mathbf{x}, \mathbf{x} \rangle \geq 0, \tag{3.5}$$

since φ is l-positive.

Conversely, let l be an arbitrary natural number and consider $\varphi_l\colon M_l \otimes M_n \to M_l \otimes M_k$. Let $A \in (M_l \otimes M_n)^+$. With the identification $M_l \otimes M_n \simeq M_l(M_n)$ as explained before, there exists $B \in M_l(M_n)$ such that $A = B^* B$. Letting $B = [B_{i,j}]_{i,j=1}^{l}$, with $B_{i,j} \in M_n$, we have

$$A = \sum_{p=1}^{l} B_j^* B_j \tag{3.6}$$

where B_j is the block $l \times l$ matrix with the j-th row exactly the j-th row of B and all the other rows filled with zeros. From here and (3.5) it follows easily that φ is l-positive. □

We use the proof of the previous proposition to derive a useful characterization of the natural order relation on $\mathrm{CP}(M_n; M_k)$.

Corollary 3.1. *Let $\varphi, \psi \in \mathrm{CP}(M_n; M_k)$. Then $\varphi \leq \psi$ if and only if for all natural numbers l, matrices $A_1, \ldots, A_l \in M_n$, and vectors $x_1, \ldots, x_l \in \mathbb{C}^k$, we have*

$$\sum_{i,j=1}^{l} \langle \varphi(A_i^* A_j) x_j, x_i \rangle \leq \sum_{i,j=1}^{l} \langle \psi(A_i^* A_j) x_j, x_i \rangle \tag{3.7}$$

3.3. The Stinespring Representation

In this subsection we prove that any completely positive map has a Stinespring representation (see Example 3.3.2).

Theorem 3.1 (W.F. Stinespring [11]) *For any $\theta \in \mathrm{CP}(M_n, M_k)$ there exists a triple $(\pi_\theta; V_\theta; \mathbb{C}^m)$ subject to the following properties:*

(st1) $m \leq n^2 k$ *is a natural number.*

(st2) $\pi_\theta \colon M_n \to M_m$ *is a morphism of $*$-algebras and V is an $m \times k$ matrix, such that $\theta(A) = V_\theta^* \pi_\theta(A) V_\theta$ for all $A \in M_n$.*

(st3) $\mathrm{Lin}(\pi_\theta(M_n) V_\theta \mathbb{C}^k) = \mathbb{C}^m$.

In addition, the triple $(\pi_\theta; V_\theta; m)$ is unique, up to a unitary (orthonormal) matrix $U \in M_m$, in the sense that if $(\pi; V; \mathbb{C}^{m'})$ is another triple subject to the conditions (st1)-(st3)*, then $m = m'$ and there exists a unitary (orthonormal) matrix $U \in M_m$ such that $\pi(A) V = U \pi_\theta(A) V_\theta$ for all $A \in M_n$.*

The triple $(\pi_\theta; V_\theta; \mathbb{C}^m)$ is called the *Minimal Stinespring Representation* of φ.

We briefly sketch the existence part in Theorem 3.1.

On the vector space $M_n \otimes \mathbb{C}^k$ we consider the inner product $\langle \cdot, \cdot \rangle_\theta$ defined as follows: for l and p natural numbers, matrices $A_1, \ldots, A_l, B_1, \ldots, B_p \in M_n$ and vectors $x_1, \ldots, x_l, y_1, \ldots, y_p \in \mathbb{C}^k$, all arbitrary, let

$$\langle \sum_{i=1}^{l} A_i \otimes x_i, \sum_{j=1}^{p} B_j \otimes y_j \rangle_\theta = \sum_{i=1}^{l} \sum_{j=1}^{p} \langle \theta(B_j^* A_i) x_i, y_j \rangle. \tag{3.8}$$

The inner product $\langle \cdot, \cdot \rangle_\theta$ is positive semidefinite by Proposition 3.1. We factor $M_n \otimes \mathbb{C}^k$ by the null space $\mathcal{N}_\theta$ of the semidefinite inner product $\langle \cdot, \cdot \rangle_\theta$

$$\mathcal{N}_\theta = \{ f \in M_n \otimes \mathbb{C}^k \mid \langle f, f \rangle_\theta = 0 \} \tag{3.9}$$

and get a vector space of dimension m, on which this inner product is positive definite. Clearly, the dimension of this new vector space $M_n \otimes \mathbb{C}^k/\mathcal{N}_\theta$ is at most n^2k and, modulo a unitary identification, without loss of generality we can take $M_n \otimes \mathbb{C}^k/\mathcal{N}_\theta = \mathbb{C}^m$.

The $*$-morphism $\pi_\theta\colon M_n \to M_m$ is firstly defined on elementary tensors by

$$\pi_\theta(A)(B \otimes x) = (AB) \otimes x, \quad A, B \in M_n,\ x \in \mathbb{C}^k, \tag{3.10}$$

and then it can be proven that $\pi(A)$ factors by $\mathcal{N}_\theta$. Then we let $V_\theta x = [I_n \otimes x]_\theta$, where for any element $h \in M_n \otimes \mathbb{C}^k$ we denoted by $[h]_\theta$ its equivalence class, modulo the factorization and its identification by the corresponding orthonormal transformation to a vector in $\mathbb{C}^m$.

Let us note that if $\theta \in \mathrm{CP}(M_n, M_k)$ is *unital*, that is, $\theta(I_n) = I_k$, then V_θ is an isometric transformation (its columns are orthonormal).

3.4. The Arveson's Radon-Nikodym Derivative

Let $\varphi, \theta \in \mathrm{CP}(M_n; M_k)$ be such that $\varphi \leq \theta$ and consider the Minimal Stinespring Representation $(\pi_\varphi; V_\varphi; \mathbb{C}^p)$ of φ, and similarly the Minimal Stinespring Representation $(\pi_\theta; V_\theta; \mathbb{C}^m)$ of θ. With the notation as in (3.9), from $\varphi \leq \theta$ and Corollary 3.1, the identity operator $J_{\varphi,\theta}\colon M_n \otimes \mathbb{C}^k \to M_n \otimes \mathbb{C}^k$ has the property that $J_{\varphi,\theta}\mathcal{N}_\theta \subseteq \mathcal{N}_\varphi$, hence it can be factored to a linear operator $J_{\varphi,\theta}\colon (M_n \otimes \mathbb{C}^k)/\mathcal{N}_\theta \to (M_n \otimes \mathbb{C}^k)/\mathcal{N}_\varphi$ and then, modulo the unitary identification of these spaces with $\mathbb{C}^m$ and, respectively, $\mathbb{C}^p$ it is a contractive linear operator $J_{\varphi,\theta} \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^p)$, that is, a contractive $p \times m$ matrix. It is easy to see that

$$J_{\theta,\varphi} V_\theta = V_\varphi, \tag{3.11}$$

and that

$$J_{\theta,\varphi}\pi_\theta(A) = \pi_\varphi(A) J_{\theta,\varphi}, \text{ for all } A \in M_n. \tag{3.12}$$

Thus, letting

$$\mathrm{D}_\theta(\varphi) := J_{\theta,\varphi}^* J_{\theta,\varphi} \tag{3.13}$$

we get a contractive linear operator in $\mathcal{L}(\mathbb{C}^m)$. In addition, as a consequence of (3.12), $\mathrm{D}_\theta(\varphi)$ commutes with all operators $\pi_\theta(A)$ for $A \in \mathcal{A}$, briefly, $\mathrm{D}_\theta(\varphi) \in \pi_\theta(\mathcal{A})'$; indeed, by taking adjoints in (3.12) we have $\pi_\theta(A) J_{\theta,\varphi}^* = J_{\theta,\varphi}^* \pi_\varphi(A)$ for all $A \in M_n$, hence

$$\begin{aligned}
\mathrm{D}_\theta(\varphi)\pi_\theta(A) &= J_{\theta,\varphi}^* J_{\theta,\varphi}\pi_\theta(A) \\
&= J_{\theta,\varphi}^* \pi_\varphi(A) J_{\theta,\varphi} = \pi_\theta(A) J_{\theta,\varphi}^* J_{\theta,\varphi} \\
&= \pi_\theta(A)\, \mathrm{D}_\theta(\varphi), \quad A \in M_n.
\end{aligned}$$

In addition, from (3.11) and (3.13) it follows

$$\varphi(A) = V_\varphi^* \pi_\varphi(A) V_\varphi = V_\theta^*\, \mathrm{D}_\theta(\varphi)\pi_\theta(a) V_\theta \text{ for all } A \in M_n,$$

which, taking into account that $D_\theta(\varphi) \in \pi(\mathcal{A})'$, and hence $D_\theta(\varphi)^{1/2} \in \pi(\mathcal{A})'$, we write

$$\varphi(A) = V_\theta^* \, \mathrm{D}_\theta(\varphi)^{1/2} \pi_\theta(A) \, \mathrm{D}_\theta(\varphi)^{1/2} V_\theta, \text{ for all } A \in M_n. \tag{3.14}$$

It is immediate from (3.14) that, for any $l \in \mathbb{N}$, $(A_j)_{j=1}^l \in M_n$, and $(h_j)_{j=1}^l \in \mathbb{C}^k$, the following formula holds

$$\sum_{i,j=1}^{l} \langle \varphi(A_j^* A_i) h_i, h_j \rangle = \| \, \mathrm{D}_\theta(\varphi)^{1/2} \sum_{j=1}^{n} \pi_\theta(A_j) V_\theta h_j \|^2. \tag{3.15}$$

It is easy to show that (3.14) is equivalent to (3.15). The property (3.14) uniquely characterizes the operator $\mathrm{D}_\theta(\varphi)$. The operator $\mathrm{D}_\theta(\varphi)$ is called the *Radon-Nikodym derivative* of φ with respect to θ.

Recalling Corollary 3.1, (3.15) shows that for any $\varphi, \psi \in \mathrm{CP}(\mathcal{A}; \mathcal{H})$ with $\varphi, \psi \leq \theta$, we have $\varphi \leq \psi$ if and only if $\mathrm{D}_\theta(\varphi) \leq \mathrm{D}_\theta(\psi)$.

In addition, if $\varphi, \psi \in \mathrm{CP}(M_n; M_k)$ are such that $\varphi, \psi \leq \theta$ then for any $t \in [0,1]$ the completely positive map $(1-t)\varphi + t\psi$ is $\leq \theta$ and

$$\mathrm{D}_\theta((1-t)\varphi + t\psi) = (1-t) \, \mathrm{D}_\theta(\varphi) + t \, \mathrm{D}_\theta(\psi). \tag{3.16}$$

The above considerations can be summarized in the following

Theorem 3.2 (W.B. Arveson [1]) *Let $\theta \in \mathrm{CP}(M_n; M_k)$. The mapping $\varphi \mapsto \mathrm{D}_\theta(\varphi)$ defined in* (3.13)*, with its inverse given by* (3.14)*, is an affine and order-preserving isomorphism between the convex and partially ordered sets $\big(\{\varphi \in \mathrm{CP}(M_n; M_k) \mid \varphi \leq \theta\}; \leq\big)$ and $\big(\{D \in \pi_\theta(M_n)' \mid 0 \leq D \leq I\}; \leq\big)$.*

One says that ψ *uniformly dominates* φ, and we write $\varphi \leq_{\mathrm{u}} \psi$, if for some $t > 0$ we have $\varphi \leq t\psi$. This is a partial preorder relation (only reflexive and transitive). The associated equivalence relation (we can call it *uniform equivalence*) is denoted by $\simeq_{\mathrm{u}}$, that is, for $\varphi, \psi \in \mathrm{CP}(M_n; M_k)$ we have $\varphi \simeq_{\mathrm{u}} \psi$ if and only if $\varphi \leq_{\mathrm{u}} \psi \leq_{\mathrm{u}} \varphi$. It is immediate from Theorem 3.2 the following

Corollary 3.2. *For a given $\theta \in \mathrm{CP}(M_n; M_k)$, the mapping $\varphi \mapsto \mathrm{D}_\theta(\varphi)$ defined in* (3.13)*, with its inverse given by* (3.14)*, is an affine and order-preserving isomorphism between the convex cones $\big(\{\varphi \in \mathrm{CP}(M_n; M_k) \mid \varphi \leq_{\mathrm{u}} \theta\}; \leq\big)$ and $\big(\{D \in \pi_\theta(M_n)' \mid 0 \leq D\}; \leq\big)$.*

4. The Kraus Form and the Choi's Matrix

In this section we focus on completely positive maps from M_n, the C^*-algebra of $n \times n$ matrices, to M_k, for which we describe the Kraus form and the Choi's matrix representation.

We first consider the system of matrix units $\{E_{i,j}^{(n,k)} \mid i=1,\ldots,n,\ j=1,\ldots,k\}$, that makes a basis of $M_{n,k}$, on which we perform a lexicographic reindexing, more precisely

$$\big(E_{1,1}^{(n,k)},\ldots,E_{1,k}^{(n,k)},E_{2,1}^{(n,k)},\ldots,E_{n,1}^{(n,k)},\ldots,E_{n,k}^{(n,k)}\big)=\big(\mathcal{E}_1,\mathcal{E}_2,....,\mathcal{E}_{nk}\big) \tag{4.1}$$

An explicit form of this reindexing is the following

$$\mathcal{E}_r=E_{i,j}^{(n,k)} \text{ where } r=(i-1)k+j, \text{ for all } i=1,\ldots,n,\ j=1,\ldots,k. \tag{4.2}$$

Proposition 4.1. *The formula*

$$\varphi_{(i-1)k+m,(j-1)k+l}=\langle\varphi(E_{i,j}^{(n)})e_l^{(k)},e_m^{(k)}\rangle,\quad m,l=1,\ldots,k,\ i,j=1,\ldots,n, \tag{4.3}$$

and its inverse

$$\varphi(C)=\sum_{r,s=1}^{nk}\varphi_{r,s}\mathcal{E}_r^*C\mathcal{E}_s,\quad C\in M_n, \tag{4.4}$$

establish a linear and bijective correspondence

$$\mathcal{L}(M_n,M_k)\ni\varphi\mapsto\Phi=[\varphi_{r,s}]_{r,s=1}^{nk}\in M_{nk}. \tag{4.5}$$

Proof. Clearly, the correspondence $M_{nk}\ni\Phi\mapsto\varphi\in\mathcal{L}(M_n,M_k)$ given by (4.4) is linear, so it remains to prove that it is bijective and that its inverse is given by the formula (4.3). To see this, let $i,j\in\{1,\ldots,n\}$ and $l,m\in\{1,\ldots,k\}$ be arbitrary. Thus, assuming that (4.4) holds, we have

$$\langle\varphi(E_{i,j}^{(n)})e_l^{(k)},e_m^{(k)}\rangle=\sum_{r,s}^{nk}\varphi_{r,s}\langle\mathcal{E}_r^*E_{i,j}^{(n)}\mathcal{E}_se_l^{(k)},e_m^{(k)}\rangle$$

and, by representing uniquely $r=(q-1)k+p$ and $s=(b-1)k+a$, for $a,p\in\{1,\ldots,k\}$ and $b,q\in\{1,\ldots,n\}$, we get

$$=\sum_{a,p=1}^{k}\sum_{b,q=1}^{n}\varphi_{(q-1)k+p,(b-1)k+a}\langle E_{i,j}^{(n)}E_{b,a}^{(n,k)}e_l^{(k)},E_{q,p}^{(n,k)}e_m^{(k)}\rangle$$

and then, by (2.14) and (2.7), we get

$$=\varphi_{(i-1)k+m,(j-1)k+l}.$$

□

Remark 4.1. With respect to the identification $\mathbb{C}^{nk}\simeq\mathbb{C}^n\otimes\mathbb{C}^k$, any matrix $\Phi=[\varphi_{r,s}]_{r,s=1}^{nk}\in M_{nk}=\mathcal{L}(\mathbb{C}^{nk})$ is identified with a linear operator $\Phi\in\mathcal{L}(\mathbb{C}^n\otimes\mathbb{C}^k)$, in such a way that the formula (4.3) becomes

$$\varphi_{(i-1)k+m,(j-1)k+l}=\langle\Phi(e_j^{(n)}\otimes e_l^{(k)}),e_i^{(n)}\otimes e_m^{(k)}\rangle,\ m,l=1,\ldots,k,\ i,j=1,\ldots,n. \tag{4.6}$$

Remark 4.2. In the correspondence in Proposition 4.1, φ is unital if and only if

$$\sum_{i=1}^{n} \varphi_{(i-1)k+m,(i-1)k+l} = \delta_{m,l} \text{ for all } l, m \in \{1, \dots, k\}.$$

Let $\rho \colon M_n \to M_k$ be the *tracial* map defined by

$$\rho(C) = \frac{1}{n} \operatorname{tr}(C) I_k, \quad C \in M_n. \tag{4.7}$$

Let the linear mapping

$$V \colon \mathbb{C}^k \to \mathbb{C}^{n^2 k} \simeq \mathbb{C}^n \otimes \mathbb{C}^{nk} \simeq \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^k$$

be defined by

$$Vh = \frac{1}{\sqrt{n}} \begin{bmatrix} \mathcal{E}_1 h \\ \mathcal{E}_2 h \\ \vdots \\ \mathcal{E}_{nk} h \end{bmatrix}, \quad h \in \mathbb{C}^k, \tag{4.8}$$

or, equivalently, with the identification $\mathbb{C}^{n^2 k} \simeq \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^k$ and the reindexing defined at (4.1),

$$Vh = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{k} E_{i,j}^{(n,k)} h \otimes e_i^{(n)} \otimes e_j^{(k)}. \tag{4.9}$$

We consider also the map

$$\pi \colon M_n \to M_{n^2 k} \simeq \mathcal{L}(\mathbb{C}^{n^2 k}) \simeq \mathcal{L}(\mathbb{C}^n \otimes \mathbb{C}^{nk}),$$

defined by

$$\pi(C) = C \otimes I_{nk}, \quad C \in M_n. \tag{4.10}$$

Proposition 4.2. *With the notation as in* (4.7)–(4.10), $(\pi; V; \mathbb{C}^{n^2 k})$ *is the Minimal Stinespring Representation of* ρ, *in particular,* $\rho \in \mathrm{CP}(M_n, M_k)$.

Proof. Clearly, π is a $*$-representation. We prove that

$$\rho(C) = V^* \pi(C) V, \quad C \in M_n. \tag{4.11}$$

Indeed, for any $i, j \in \{1, \dots, n\}$

$$V^* \pi(E_{i,j}^{(n)}) V = \frac{1}{n} \sum_{l=1}^{nk} \mathcal{E}_l^* E_{i,j}^{(n)} \mathcal{E}_l = \frac{1}{n} \sum_{r=1}^{n} \sum_{s=1}^{k} E_{r,s}^{(n,k)*} E_{i,j}^{(n)} E_{r,s}^{(n,k)}$$

which, taking into account of (2.6), becomes

$$= \frac{1}{n} \sum_{r=1}^{n} \sum_{s=1}^{k} E_{s,s}^{(k,n)} E_{i,j}^{(n)} E_{r,s}^{(n,k)}$$

then, taking into account of (2.7), we get

$$\begin{aligned} &= \frac{1}{n} \sum_{r=1}^{n} \sum_{s=1}^{k} \delta_{r,i} \delta_{j,r} E_{s,s}^{(k)} \\ &= \frac{1}{n} \sum_{s=1}^{k} \Big(\sum_{r=1}^{n} \delta_{r,i} \delta_{j,r} \Big) E_{s,s}^{(k)} \\ &= \frac{1}{n} \delta_{i,j} I_k = \frac{1}{n} \operatorname{tr}(E_{i,j}^{(n)}) I_k = \rho(E_{i,j}^{(n)}). \end{aligned}$$

Since $\{E_{i,j}^{(n)} \mid i,j = 1, \ldots, n\}$ is a linear basis of M_n, this proves (4.11).

It remains to prove the minimality condition, that is, that

$$\mathbb{C}^{n^2 k} = \operatorname{Lin}\{\varphi(M_n) V \mathbb{C}^k\}. \tag{4.12}$$

To see this, let $i, j \in \{1, \ldots, n\}$ and $m \in \{1, \ldots, k\}$ be arbitrary. Then

$$(E_{i,j}^{(n)} \otimes I_{nk}) V e_m^{(k)} = (E_{i,j}^{(n)} \otimes I_n \otimes I_k) V e_m^{(k)}$$

which, taking into account of (4.9), becomes

$$\begin{aligned} &= \sum_{r=1}^{n} \sum_{s=1}^{k} (E_{i,j}^{(n)} \otimes I_n \otimes I_k)(E_{r,s}^{(n,k)} e_m^{(k)} \otimes e_r^{(n)} \otimes e_s^{(k)}) \\ &= \sum_{r=1}^{n} \sum_{s=1}^{k} (E_{i,j}^{(n)} E_{r,s}^{(n,k)} e_m^{(k)} \otimes e_r^{(n)} \otimes e_s^{(k)}) \end{aligned}$$

then, taking into account of (2.7), we get

$$= \sum_{s=1}^{n} E_{i,s}^{(n,k)} e_m^{(k)} \otimes e_j^{(n)} \otimes e_s^{(k)}$$

which, taking into account that $E_{i,s}^{(n,k)} e_m^{(k)} = \delta_{m,s} e_i^{(n)}$, becomes

$$= e_i^{(n)} \otimes e_j^{(n)} \otimes e_m^{(k)}.$$

Since $\{e_i^{(n)} \otimes e_j^{(n)} \otimes e_m^{(k)} \mid i,j = 1, \ldots, n,\ m = 1, \ldots, k\}$ is a basis for $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^k = \mathbb{C}^{n^2 k}$, the proof of (4.12) is complete. Thus, $(\pi; V; \mathbb{C}^{n^2 k})$ is the Minimal Stinespring Representation of ρ.

Since ρ is unital, note that actually V is an embedding of $\mathbb{C}^k$ into $\mathbb{C}^{n^2k}$, in agreement with the requirements of the Minimal Stinespring Representation for this particular case. $\square$

Proposition 4.3. *The tracial map ρ uniformly dominates any map $\varphi \in \mathrm{CP}(M_n, M_k)$.*

Proof. We prove that any linear map $\varphi \in \mathrm{CP}(M_k, M_n)$ is uniformly dominated by ρ, that is, there exists $t > 0$ such that, for all $m \in \mathbb{N}$, $(a_j)_{j=1}^m \subset M_n$, and all $(h_j)_{j=1}^m \in \mathbb{C}^n$, we have

$$\sum_{i,j=1}^{m} \langle \varphi(a_j^* a_i) h_i, h_j \rangle \leq t \sum_{i,j=1}^{m} \langle \rho(a_j^* a_i) h_i, h_j \rangle. \tag{4.13}$$

To see this, note that the left side of (4.13) represents the inner product $\langle \cdot, \cdot \rangle_\varphi$ on $M_n \otimes \mathbb{C}^k$ as in (3.8), and similarly, the sum in the right hand side of (4.13) represents the inner product $\langle \cdot, \cdot \rangle_\rho$ on $M_n \otimes \mathbb{C}^k$ as in (3.8). On the other hand, due to the minimality property (4.12), it follows that the inner product $\langle \cdot, \cdot \rangle_\rho$ is nondegenerate and hence, that the associated seminorm $\| \cdot \|_\rho$ is actually a norm. Since $M_n \otimes \mathbb{C}^k$ has finite dimension, any seminorm, in particular, $\| \cdot \|_\varphi$, is $\| \cdot \|_\rho$-continuous, and hence (4.13) holds for some $t > 0$. $\square$

Theorem 4.1 (K. Kraus [6]) *Let $\varphi : M_n \to M_k$ be a completely positive map. Then there are $n \times k$ matrices $V_1, V_2, \ldots, V_m$ with $m \leq nk$ such that*

$$\varphi(A) = V_1^* A V_1 + V_2^* A V_2 + \cdots + V_m^* A V_m \text{ for all } A \in M_n. \tag{4.14}$$

Proof. To see this, we consider ρ and its Minimal Stinespring Representation $(\pi, V, \mathbb{C}^{n^2k})$ as in Proposition 4.2. Since, by Proposition 4.3 ρ uniformly dominates φ, we can apply Theorem 3.2 and get $\mathrm{D}_\rho(\varphi) \geq 0$ in the commutant of $\pi(M_n)$ such that $\varphi = V^* \mathrm{D}_\rho(\varphi)^{1/2} \pi(\cdot) \rho_\theta(\varphi)^{1/2} V$. By considering $n^2k \times n^2k$ matrices as $nk \times nk$ block matrices we see that

$$\mathrm{D}_\rho(\varphi)^{1/2} V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_{nk} \end{pmatrix}$$

for some $n \times k$ matrices $V_1, V_2, ..., V_{nk}$. Since $\varphi = (\mathrm{D}_\rho(\varphi)^{1/2} V)^* \pi(\cdot) \mathrm{D}_\rho(\varphi)^{1/2} V$ and $\pi(A)$ is the diagonal block matrix with A's on the diagonal, (4.14) follows. $\square$

The main result of this section is the following description of completely positive maps in terms of Choi's matrices.

Theorem 4.2 (M.-D. Choi [3]) *The formulae* (4.3) *and its inverse* (4.4) *establish an affine and order preserving isomorphism*

$$\mathrm{CP}(M_n, M_k) \ni \varphi \mapsto \Phi \in M_{nk}^+. \tag{4.15}$$

Proof. Consider the completely positive map $\rho\colon M_n \to M_k$ defined at (4.7), as well as its Minimal Stinespring Representation $(\pi; V; \mathbb{C}^{n^2k})$, as proven in Proposition 4.2. We combine the facts obtained so far in Proposition 4.1, Proposition 4.2, and Proposition 4.3 with those in Corollary 3.2 in order to get that the Radon-Nikodym derivative with respect to ρ establishes an affine and order preserving isomorphism between the cones

$$\mathrm{CP}(M_n, M_k) \ni \varphi \mapsto \mathrm{D}_\rho(\varphi) \in \pi(M_n)'^{+}. \tag{4.16}$$

Since

$$\pi(M_n)' = \big(M_n \otimes I_{nk}\big)' = I_n \otimes M_{nk},$$

and this identification induces an affine and order preserving isomorphism between the corresponding cones of positive elements, it follows that the Radon-Nikodym derivative with respect to ρ establishes an affine and order preserving isomorphism

$$\mathrm{CP}(M_n, M_k) \ni \varphi \mapsto \Phi \in M_{nk}^+,$$

more precisely

$$\mathrm{D}_\rho(\varphi) = I_n \otimes \Phi, \quad \varphi \in \mathrm{CP}(M_n, M_k). \tag{4.17}$$

It remains to prove that the isomorphism (4.17) coincides with that defined at (4.15), which, by the uniqueness of the Radon-Nikodym derivative, is equivalent with proving that

$$\varphi(C) = V^*(I_n \otimes \Phi)(C \otimes I_{nk})V = V^*(C \otimes \Phi)V. \tag{4.18}$$

To see this, it is sufficient to prove that for all $i, j \in \{1, \ldots, n\}$ and all $l, m \in \{1, \ldots, k\}$ we have

$$\langle \varphi(E_{i,j}^{(n)}) e_l^{(k)}, e_m^{(k)} \rangle = \langle V^*(E_{i,j}^{(n)} \otimes \Phi) V e_l^{(k)}, e_m^{(k)} \rangle. \tag{4.19}$$

First, we note that

$$\begin{aligned} V e_l^{(k)} &= \sum_{r=1}^{n} \sum_{s=1}^{k} (E_{r,s}^{(n,k)} e_l^{(k)}) \otimes e_r^{(n)} \otimes e_s^{(k)} \\ &= \sum_{r=1}^{n} \sum_{s=1}^{k} \delta_{l,s} e_r^{(n)} \otimes e_r^{(n)} \otimes e_s^{(k)} \\ &= \sum_{r=1}^{n} e_r^{(n)} \otimes e_r^{(n)} \otimes e_l^{(k)}. \end{aligned}$$

Then

$$
\begin{aligned}
\langle V^*(E_{i,j}^{(n)} \otimes \Phi) V e_l^{(k)}, e_m^{(k)} \rangle = \\
&= \langle (E_{i,j}^{(n)} \otimes \Phi) V e_l^{(k)}, V e_m^{(k)} \rangle \\
&= \sum_{r=1}^{n} \sum_{p=1}^{n} \langle (E_{i,j}^{(n)} \otimes \Phi)(e_r^{(n)} \otimes e_r^{(n)}) \otimes e_l^{(k)}, e_p^{(n)} \otimes e_p^{(n)} \otimes e_m^{(k)} \rangle \\
&= \sum_{r=1}^{n} \sum_{p=1}^{n} \langle \delta_{r,j} e_i^{(n)} \otimes (e_r^{(n)} \otimes e_l^{(k)}), e_p^{(n)} \otimes e_p^{(n)} \otimes e_m^{(k)} \rangle \\
&= \sum_{p=1}^{n} \langle e_i^{(n)} \otimes \big(\Phi(e_j^{(n)} \otimes e_l^{(n)}), e_p^{(n)} \otimes e_p^{(n)} \otimes e_m^{(k)} \rangle \\
&= \sum_{p=1}^{n} \langle e_i^{(n)}, e_p^{(n)} \rangle \langle \Phi(e_j^{(n)} \otimes e_l^{(n)}), e_p^{(n)} \otimes e_m^{(k)} \rangle \\
&= \sum_{p=1}^{n} \delta_{j,p} \langle \Phi(e_j^{(n)} \otimes e_l^{(n)}), e_p^{(n)} \otimes e_m^{(k)} \rangle \\
&= \langle \Phi(e_j^{(n)} \otimes e_l^{(k)}, e_i^{(n)} \otimes e_m^{(k)} \rangle \\
&= \langle \varphi(E_{i,j}^{(n)}) e_l^{(k)}, e_m^{(k)} \rangle,
\end{aligned}
$$

where, at the last step, we used (4.6) and (4.3). Thus, (4.19) is proven, and hence (4.18) is proven. ◻

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