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A generalization of convergent series

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To Professor Ion Colojoară on the occasion of his 80th birthday

Abstract - The notion of convergent series is generalized, using functional analysis techniques. The concept thus obtained, called here S-convergent series, is analyzed and illustrated by means of many examples.

Key words and phrases : convergent series, Banach space, linear and continuous operator, Riesz space, L^p-space, Banach algebra, spectrum of an operator, projection.

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1. Introduction

Throughout the paper $\mathbb{R}$ will denote the set of real numbers, $\mathbb{C}$ the set of complex numbers and $\mathbb{N}$ the set of natural numbers. We shall write K to denote $\mathbb{R}$ or $\mathbb{C}$.

For any two normed spaces $(X, \|\|)$, $(Y, ||| \; |||)$ and any linear and continuous operator $T : X \longrightarrow Y$, the (operator) norm of T will be

$$\|T\|_o = \sup\{|||T(x)||| \mid x \in X, \|x\| \leq 1\}.$$

We shall be concerned with the following linear spaces:

1) The space of sequences which converge to zero:

$$c_0 = \{x = (x_0, x_1, ..., x_n, ...) \mid x_n \in K,\, x_n \to 0\}$$

which is a Banach space when equipped with the usual norm

$$\|x\| = \sup_n |x_n|\,.$$

2) The space of convergent series. This is the subspace of c_0 defined as follows:

$$cc_0 = \left\{x = (x_0, x_1, ..., x_n, ...) \in c_0 \mid \sum_{n=0}^{\infty} x_n \text{ is convergent}\right\}.$$

3) The space of absolutely convergent series (the space of summable sequences). This is the subspace of c_0 defined as follows:

$$l^1 = \left\{ x = (x_0, x_1, ..., x_n, ...) \in c_0 \mid \sum_{n=0}^{\infty} |x_n| \text{ is convergent} \right\}.$$

4) The space of continuous functions. This is the space defined as follows:

One considers two real numbers a, b such that $a < b$. We obtain the space

$$C[a,b] = \{ f : [a,b] \to K \mid f \text{ is continuous} \}$$

which becomes a Banach space when equipped with the norm

$$\|f\| = \sup\{ |f(t)| \mid t \in [a,b] \}.$$

5) The space $L^p(\mu)$ which is defined as follows:

One considers a number p such that $1 \leq p < \infty$. Let us denote by μ the Lebesgue measure on $[0,1]$.

We obtain the vector space

$$\mathcal{L}^p(\mu) = \{ f : [0,1] \to K \mid f \text{ is } \mu\text{-measurable and } |f|^p \text{ is } \mu\text{-integrable} \}$$

which is seminormed with the seminorm

$$N_p(f) = \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

The null space of $\mathcal{L}^p(\mu)$ is

$$\mathcal{N}(\mu) = \{ f \in \mathcal{L}^p(\mu) \mid N_p(f) = 0 \} = \{ f : [0,1] \to K \mid f(t) = 0 \ \mu\text{-a.e.} \}.$$

The quotient space

$$L^p(\mu) \stackrel{def}{=} \mathcal{L}^p(\mu) / \mathcal{N}(\mu)$$

is a Banach space, when equipped with the (quotient) norm

$$\left\| \widetilde{f} \right\|_p \stackrel{def}{=} N_p(f)$$

for any representative $f \in \widetilde{f} \in L^p(\mu)$.

For general Analysis see [7]. For general Functional Analysis see [3], [4], [6] and [8]. For Measure Theory see [1], [2] and [5]. For Spectral Theory see [3] and [6].

2. Results

We start with a remark which motivates the subsequent facts.

Let us consider the numerical series

$$\sum_{n=0}^{\infty} x_n \tag{2.1}$$

with terms in $K = \mathbb{R}$ or $\mathbb{C}$. Cauchy's criterion asserts that the convergence of (2.1) is equivalent to the following fact: for any $\varepsilon > 0$, there exists a natural $p(\varepsilon)$ such that for any natural $p \geq p(\varepsilon)$ and any natural n, one has

$$|x_p + x_{p+1} + ... + x_{p+n}| < \varepsilon. \tag{2.2}$$

In order to give an alternative expression for (2.2), we introduce the shift operator $S : c_0 \to c_0$ (which is linear, continuous and $\|S\|_o = 1$) given via

$$S((x_0, x_1, x_2, ..., x_n, ...)) = (x_1, x_2, ..., x_n, ...).$$

Let us write $S^m = S \circ S \circ ... \circ S$ (m times) and $S^0 = I =$ the identity operator of c_0. Now, (2.2) becomes: for any $\varepsilon > 0$, there exists a natural $p(\varepsilon)$ such that for any natural $p \geq p(\varepsilon)$ and any natural n, one has

$$\|S^p(x^n)\| < \varepsilon. \tag{2.3}$$

Here, for $x = (x_0, x_1, x_2, ..., x_n, ...) \in c_0$ from (2.1), we write

$$x^n = x + S(x) + S^2(x) + ... + S^n(x),\ x^0 = x.$$

Taking into account the structure of (2.2) and (2.3), we have the following result for $x \in c_0$:

$$x \in cc_0 \Longleftrightarrow \lim_p S^p(x^n) = 0$$

uniformly with respect to $n \in \mathbb{N}$.

Having this in mind, we shall (from now on) consider a Banach space over K (real or complex) equipped with the norm $x \to \|x\|$. We shall also consider a linear and continuous operator $S : X \to X$ and $I : X \to X$ is the identity operator. For an element $x \in X$, we shall write $x^0 = x$ and, for natural $n \geq 1$:

$$x^n = x + S(x) + S^2(x) + ... + S^n(x) = (I + S + S^2 + ... + S^n)(x),$$

where

$$S^n = S \circ S \circ ... \circ S \ (n \text{ times}),$$

$$S^0 = I.$$

Definition 2.1. *An element $x \in X$ is called S-convergent series if it has the property that*

$$\lim_p S^p(x^n) = 0 \tag{2.4}$$

uniformly with respect to $n \in \mathbb{N}$. Here (2.4) means: for any $\varepsilon > 0$, there exists a natural $p(\varepsilon)$, such that for any natural $p \geq p(\varepsilon)$ and any natural n, one has

$$\|S^p(x^n)\| < \varepsilon.$$

The set of all S-convergent series will be denoted by $C(S)$.

It is easily seen that $C(S)$ is a linear subspace of X and $C(S) \supset Ker(S)$.

Theorem 2.1. *For an element $x \in X$, the following assertions are equivalent:*

1. *One has $x \in C(S)$.*
2. *The series $\sum\limits_{p=0}^{\infty} S^p(x)$ is convergent (in X).*

Proof. The series in the statement is convergent if and only if the sequence $\left(\sum\limits_{p=0}^{m} S^p(x)\right)_m$ is Cauchy. This is equivalent to the following fact: for any $\varepsilon > 0$, there exists $p(\varepsilon) \in \mathbb{N}$, such that, for any natural $p \geq p(\varepsilon)$ and any natural $n \geq 1$, one has

$$\left\| \sum_{m=0}^{p+n} S^m(x) - \sum_{m=0}^{p} S^m(x) \right\| = \left\| S^{p+1}(x^{n-1}) \right\| < \varepsilon.$$

The last assertion is precisely the assertion that

$$\lim_p S^p(x^n) = 0$$

uniformly with respect to $n \in \mathbb{N}$. □

Using Theorem 2.1 we get

Proposition 2.1. *For an element $x \in X$, the following assertions are equivalent:*

1. *One has $x \in C(S)$.*
2. *There exists a natural p such that $S^p(x) \in C(S)$.*
3. *For any natural p one has $S^p(x) \in C(S)$.*

Consequently, $C(S)$ is a linear subspace of X, which is invariant with respect to S (and, of course, $Ker(S) \subset C(S)$). We shall see (Example 3.1) that, generally speaking, $C(S)$ can be not closed.

How large is $C(S)$? The following theorem answers this question.

Theorem 2.2. *Either $C(S) = X$ or $C(S)$ is of the first Baire category (i.e. $C(S)$ is meager).*

Proof. Let us assume that $C(S) \neq X$. We must prove that $C(S)$ is of the first category. Accepting the contrary, we shall arrive at a contradiction.

So let us assume that $C(S)$ is of the second category. We construct the sequence of linear and continuous operators $(T_n)_n$ given via

$$T_n = I + S + S^2 + ... + S^n$$

with $T_0 = I$. According to the definition of $C(S)$, the sequence $(T_n)_n$ converges pointwise on $C(S)$ which is of the second category. Using the Banach-Steinhaus theorem we get the fact that the sequence $(T_n)_n$ converges pointwise on all of X and the pointwise limit is a linear and continuous operator (see [8], p. 84 Corollary and p. 86 Corollary or see [3]). In other words, the series $\sum\limits_{n=0}^{\infty} S^n(x)$ converges for all $x \in X$, hence $C(S) = X$, contradiction. □

In order to continue, we define the linear map $T_S : C(S) \to X$, given via

$$T_S(x) = \sum_{n=0}^{\infty} S^n(x).$$

Theorem 2.3. *1. The map T_S is injective.*

2. For any $x \in C(S)$, one has

$$(I - S)(T_S(x)) = x.$$

Consequently

$$C(S) = (I - S)(T_S(C(S))) \subset (I - S)(X).$$

3. The following assertions are equivalent:

a) One has

$$(I - S)(X) = C(S).$$

b) For any $x \in X$ there exists the limit

$$\lim_n S^n(x).$$

In this case, for any $x \in X$ one has

$$T_S((I - S)(x)) = x - \lim_n S^n(x).$$

4. The following assertions are equivalent:

a) The map T_S is a bijection.

b) For any $x \in X$ one has

$$\lim_n S^n(x) = 0.$$

In this case, considering the map $(I-S)_1 : X \to C(S)$, given via (see 3.)

$$(I-S)_1(x) = (I-S)(x),$$

one has

$$(I-S)_1 = T_S^{-1}.$$

Proof. 1. Let $x \in C(S)$ such that

$$T_S(x) = 0.$$

So

$$0 = T_S(x) = x + S(x + S(x) + S^2(x) + ... + S^n(x) + ...) = x + S(T_S(x)) = x + 0,$$

hence

$$x = 0.$$

2. Let $x \in C(S)$.
Again

$$T_S(x) = x + S(T_S(x)),$$

hence

$$x = T_S(x) - S(T_S(x)) = (I-S)(T_S(x)).$$

We proved that any $x \in C(S)$ has the form

$$x = (I-S)(T_S(x)),$$

hence

$$C(S) \subset (I-S)(T_S(C(S))).$$

Conversely, let $y \in (I-S)(T_S(C(S)))$. We can find $x \in C(S)$ such that

$$y = (I-S)(T_S(x)) = x$$

and the inclusion

$$(I-S)(T_S(C(S))) \subset C(S)$$

is also true.

3. a)$\Rightarrow$b). In view of 2., hypothesis a) means that

$$(I-S)(X) \subset C(S).$$

Let $x \in X$ be arbitrary. Because $(I-S)(x) \in C(S)$, we can compute

$$T_S(x - S(x)) = \lim_n \left[(x - S(x)) + S(x - S(x)) + S^2(x - S(x)) + ... + S^n(x - S(x))\right] =$$

$$= \lim_n (x - S^{n+1}(x)),$$

hence $\lim_n S^n(x)$ exists and one has

$$T_S((I-S)(x)) = x - \lim_n S^n(x). \tag{2.5}$$

b)⇒a). We must prove that

$$(I-S)(X) \subset C(S).$$

Let us choose $x \in X$. Using the hypothesis, there exists

$$\lim_n \Big(x - S^{n+1}(x) \Big).$$

But we have seen that

$$x - S^{n+1}(x) = (x - S(x)) + S(x - S(x)) + S^2(x - S(x)) + ... + S^n(x - S(x))$$

which implies that the series

$$\sum_{n=0}^{\infty} S^n(x - S(x))$$

is convergent and so

$$x - S(x) \in C(S).$$

The equality in the statement is given by (2.5).

4. a)⇒b). Let $y \in X$. We must prove that

$$\lim_p S^p(y) = 0.$$

To this end, let us take an arbitrary $\varepsilon > 0$. Because T_S is surjective, we find $x \in C(S)$ such that $y = T_S(x)$. Hence, for any $p \in \mathbb{N}$, one has

$$S^p(y) = S^p(T_S(x)) = \lim_n (S^p(x) + S^{p+1}(x) + ... + S^{p+n}(x)).$$

Because $x \in C(S)$, the series

$$\sum_{n=0}^{\infty} S^n(x)$$

is convergent.

Using Cauchy's criterion, we can find a natural $p(\varepsilon)$ such that, for any natural $p \geq p(\varepsilon)$ and any natural n, one has

$$\left\| S^p(x) + S^{p+1}(x) + ... + S^{p+n}(x) \right\| < \varepsilon.$$

Let us fix such a $p \geq p(\varepsilon)$. It is possible to pass to n-limit in the last inequality and we find

$$\|S^p(y)\| = \lim_n \left\|S^p(x) + S^{p+1}(x) + ... + S^{p+n}(x)\right\| \leq \varepsilon$$

proving that

$$\lim_p S^p(y) = 0.$$

b)⇒a). One must prove that T_S (which is injective, see 1.) is surjective.

Let $x \in X$. According to 3. one has

$$(I - S)(X) = C(S)$$

and

$$y = (I - S)(x) \in C(S).$$

Again 3. says that

$$T_S(y) = T_S((I - S)(x)) = x - \lim_n S^n(x) = x.$$

Hence

$$x = T_S(y)$$

and T_S is surjective.

Now, accepting that T_S is bijective, we use again 3. and notice that, for any $x \in X$, one has

$$T_S \circ (I - S)_1(x) = x - \lim_n S^n(x) = x.$$

On the other hand, we can rephrase 2. as follows: for any $x \in C(S)$, one has

$$(I - S)_1 \circ T_S(x) = x.$$

The last two equalities show that

$$(I - S)_1 = T_S^{-1}.$$

□

The results obtained at point 4. can be rephrased as follows (we work for $K = \mathbb{C}$):

Corollary 2.1. *Assume that for any $x \in S$ one has*

$$\lim_n S^n(x) = 0.$$

Then, we have three possibilities:

1. If $C(S) = X$ (i.e. $(I-S)(X) = X$) it follows that 1 is in the resolvent set of S (and $T_S = (I - S)^{-1}$).

2. If $C(S) \neq X$, but $\overline{C(S)} = X$, it follows that 1 is in the continuous spectrum of S (and the map $T_S : C(S) \to X$ is not continuous).

3. If $C(S) \neq X$ and $\overline{C(S)} \neq X$, it follows that 1 is in the residual spectrum of S.

Using the previous result for operators on finite dimensional spaces, we obtain

Corollary 2.2. *Let us assume that X is a finite dimensional vector space over $\mathbb{C}$. Let also $S : X \to X$ be a linear operator.*

The following assertions are equivalent:

1) One has

$$\lim_n S^n(x) = 0,$$

for all $x \in X$.

2) For any $x \in X$, the series

$$\sum_{n=0}^{\infty} S^n(x)$$

is convergent.

Proof. It is clear that 2)⇒1). In order to prove 1)⇒2) we use Corollary 2.1. Possibilities 2. and 3. cannot happen, because the continuous spectrum and the residual spectrum of S are empty. It follows that possibility 1 is valid, consequently $C(S) = X$, which is precisely 2). □

Before passing further, we think it will be useful to see how Theorem 2.3 works in a particular case.

Example 2.1. We consider a non null Banach space X and a number $\alpha \in K$. The operator $S : X \to X$ will be given via

$$S(x) = \alpha x.$$

a) It is seen that

$$C(S) = \{x \in X \mid \text{there exists} \lim_n (1 + \alpha + \alpha^2 + ... + \alpha^n)x\}.$$

Hence

$$C(S) = \begin{cases} \{0\}, & \text{if } |\alpha| \geq 1 \\ X, & \text{if } |\alpha| < 1 \end{cases}.$$

b) $T_S : C(S) \to X$ is given via

$$T_S(x) = \begin{cases} 0, & \text{if } |\alpha| \geq 1 \\ \frac{1}{1-\alpha}x, & \text{if } |\alpha| < 1 \end{cases}.$$

The equality (valid for $x \in C(S)$)

$$(I - S)(T_S(x)) = x$$

is readily checked.

c) For any $\alpha \in K$, one has

$$(I-S)(X) = (1-\alpha)(X) \supset C(S).$$

For $|\alpha| < 1$, one has $(1-\alpha)(X) = X$. For $|\alpha| \geq 1$, $\alpha \neq 1$, one has $(1-\alpha)(X) = X$. Finally, for $\alpha = 1$, one has $C(S) = \{0\}$ and $(1-\alpha)(X) = \{0\}$.

d) For any $x \in X$ and $n \in \mathbb{N}$, one has

$$S^n(x) = \alpha^n x.$$

The limit

$$\lim_n \alpha^n x$$

exists for any $x \in X$ if and only if $|\alpha| < 1$ or $\alpha = 1$. It is seen that $(1-\alpha)(X) = C(S)$ in all these cases:

$$(1-\alpha)(X) = \begin{cases} X = C(S), & \text{if } |\alpha| < 1 \\ \{0\} = C(S), & \text{if } \alpha = 1 \end{cases}.$$

The equality (for $|\alpha| < 1$ or $\alpha = 1$)

$$T_S((I-S)(x)) = x - \lim_n S^n(x),$$

valid for all $x \in X$, is readily checked.

Namely, in case $|\alpha| < 1$ one has

$$T_S((I-S)(x)) = \frac{1}{1-\alpha}(1-\alpha)x = x = x - \lim_n \alpha^n x$$

and in case $\alpha = 1$ one has

$$T_S((I-S)(x)) = T_S(x-x) = T_S(0) = 0 = x - \lim_n 1^n x.$$

In case $\alpha = 1$ one can see that $T_S : \{0\} \to X$ is not a bijection.

e) One has

$$\lim_n S^n(x) = 0$$

for any $x \in X$ if and only if $|\alpha| < 1$. In this case $C(S) = X$ and $T_S : X \to X$ is the bijection given via

$$T_S(x) = \frac{1}{1-\alpha}x.$$

In the rest of the paragraph we shall work for $K = \mathbb{R}$ and we shall make the supplementary assumptions that X is a Riesz space and S is a positive operator. One can see that the real space c_0 and the shift operator $S : c_0 \to c_0$ fulfill these assumptions.

Definition 2.2. *An element $x \in X$ is called an S-absolutely convergent series in case $|x| \in C(S)$.*

The set of S-absolutely convergent series will be denoted by $AC(S)$.

Theorem 2.4. *The set $AC(S)$ is a linear subspace of $C(S)$ which is invariant with respect to S.*

Generally speaking, the inclusion $AC(S) \subset C(S)$ is strict and $AC(S)$ may not be closed (see Example 3.1).

Proof. a) We prove the inclusion

$$AC(S) \subset C(S).$$

Let $x \in AC(S)$. For an arbitrary $\varepsilon > 0$, we can find a natural $p(\varepsilon)$ such that, for any natural $p \geq p(\varepsilon)$ and any natural n, one has

$$\|S^p(|x|^n)\| < \varepsilon.$$

We have successively, for natural p and n:

$$|S^p(x^n)| = \left|S^p(x + S(x) + S^2(x) + ... + S^n(x))\right| \leq$$

$$\leq S^p(|x| + |S(x)| + \left|S^2(x)\right| + ... + |S^n(x)|) \leq$$

$$\leq S^p(|x| + S(|x|) + S^2(|x|) + ... + S^n(|x|)) = S^p(|x|^n).$$

Hence, for $p \geq p(\varepsilon)$ and $n \in \mathbb{N}$ we get

$$\|S^p(x^n)\| = \| \ |S^p(x^n)| \ \| \leq \|S^p(|x|^n)\| < \varepsilon.$$

b) It is immediate that $x \in AC(S) \Rightarrow S^p(x) \in AC(S)$ for any $p \in \mathbb{N}$, because, if $q \in \mathbb{N}$, one has

$$S^q(|S^p(x)|) \leq S^q(S^p(|x|)) = S^{q+p}(|x|)$$

which implies

$$\left\|S^q(|S^p(x)|) + S^{q+1}(|S^p(x)|) + ... + S^{q+n}(|S^p(x)|)\right\| \leq$$

$$\leq \left\|S^{p+q}(|x|) + S^{p+q+1}(|x|) + ... + S^{p+q+n}(|x|)\right\|.$$

c) If x, y are in $AC(S)$, we shall prove that $x + y \in AC(S)$.

Indeed, for any natural n:

$$|x + y|^n = |x + y| + S(|x + y|) + S^2(|x + y|) + ... + S^n(|x + y|) \leq$$

$$\leq |x|+|y|+S(|x|)+S(|y|)+S^2(|x|)+S^2(|y|)+...+S^n(|x|)+S^n(|y|) = |x|^n+|y|^n.$$

This implies, for natural p and n:

$$S^p(|x+y|^n) \leq S^p(|x|^n) + S^p(|y|^n) \text{ a.s.o.}$$

Now, if x is in $AC(S)$ and $\alpha \in \mathbb{R}$, we shall prove that $\alpha x \in AC(S)$.
Indeed, for any natural n:

$$|\alpha x|^n = |\alpha| \, |x|^n$$

and for any natural p and n:

$$S^p(|\alpha x|^n) = |\alpha| \, S^p(|x|^n) \text{ a.s.o.}$$

□

Theorem 2.5. *(Generalized D'Alembert Criterion) Let $x \in X$. Assume there exists $0 < a < 1$ such that there exists $M \in \mathbb{N}$ having the property*

$$\left|S^{n+1}(x)\right| \leq a \left|S^n(x)\right|$$

for all $n \geq M$.

Then $x \in C(S)$.

Proof. Let us take $n \geq M$. Then we have successively:

$$\left|S^{n+1}(x)\right| \leq a \left|S^n(x)\right| \Rightarrow \left\|S^{n+1}(x)\right\| \leq a \left\|S^n(x)\right\|$$

$$\left|S^{n+2}(x)\right| \leq a \left|S^{n+1}(x)\right| \leq a^2 \left|S^n(x)\right| \Rightarrow \left\|S^{n+2}(x)\right\| \leq a^2 \left\|S^n(x)\right\|$$

$$...$$

$$\left|S^{n+p}(x)\right| \leq a^p \left|S^n(x)\right| \Rightarrow \left\|S^{n+p}(x)\right\| \leq a^p \left\|S^n(x)\right\|.$$

Hence, one has

$$\begin{gathered}\left\|S^n(x)\right\| + \left\|S^{n+1}(x)\right\| + \left\|S^{n+2}(x)\right\| + ... + \left\|S^{n+p}(x)\right\| \leq \\ \leq (1 + a + a^2 + ... + a^p) \left\|S^n(x)\right\|\end{gathered}$$

and the series $\sum_{n=0}^{\infty} S^n(x)$ converges absolutely. □

In order to introduce the next results, it will be necessary to supplement our assumptions with the following ones:

A1. The space X is an algebra, with multiplication

$$(x, y) \to xy$$

and with the property that, for any x, y in X one has

$$|xy| \leq \|x\| \, |y|$$

and

$$|xy| \leq \|y\| \, |x| \, .$$

This implies

$$\|xy\| \leq \|x\| \, \|y\| \, ,$$

hence X is a Banach algebra.

A2. The operator $S : X \to X$ is multiplicative, i.e.

$$S(xy) = S(x)S(y)$$

for any x, y in X.

Consequently, S is an algebra morphism.

A3. For any $x \in X$ one has

$$\lim_{n} S^n(x) = 0.$$

One can see that the real space $X = c_0$, with natural multiplication

$$((x_0, x_1, ..., x_n, ..), (y_0, y_1, ..., y_n, ..)) \to (x_0y_0, x_1y_1, ..., x_ny_n, ..)$$

and the shift operator $S : c_0 \to c_0$ satisfy assumptions A1, A2 and A3.

Theorem 2.6. *(Generalized Abel-Dirichlet Criterion) Assume A1, A2 and A3 are fulfilled. Then, for any $x \in C(S)$ and any $a \in X$ with the property $a \geq S(a)$, one has $ax \in C(S)$ and $xa \in C(S)$.*

Proof. We begin with an algebraic property.

Let A be a ring and $a_0, a_1, ..., a_{n+1}$; $b_0, b_1, ..., b_{n+1}$, $n \geq 0$, elements in A. Then we have the identities

$$\begin{gathered} a_0b_0 + a_1b_1 + ... + a_nb_n = \\ = \sum_{i=0}^{n} (a_i - a_{i+1})(b_0 + b_1 + ... + b_i) + a_{n+1}(b_0 + b_1 + ... + b_n) \end{gathered} \tag{2.6}$$

and

$$\begin{gathered} a_0b_0 + a_1b_1 + ... + a_nb_n = \\ = \sum_{i=0}^{n} (a_0 + a_1 + ... + a_i)(b_i - b_{i+1}) + (a_0 + a_1 + ... + a_n)b_{n+1}. \end{gathered} \tag{2.7}$$

The proof can be done by induction with respect to n.

We shall take $x \in C(S)$ and $a \in X$ such that $a \geq S(a)$ and we shall prove that $ax \in C(S)$, using (2.6). The fact that $xa \in C(S)$ can be proved in a similar way, using (2.7).

In order to make things shorter, we shall write

$$ax + S(ax) + S^2(ax) + ... + S^p(ax) = \sum(p),$$

for any natural p.

Let $\varepsilon > 0$. We shall find a natural $p(\varepsilon)$ such that for any natural $n > m \geq p(\varepsilon)$ one has

$$\left\| \sum(n) - \sum(m) \right\| < \varepsilon \tag{$\star$}$$

and this will prove the convergence of the series

$$\sum_{n=0}^{\infty} S^n(ax)$$

which means $ax \in C(S)$.

Using (2.6) and the fact that

$$\sum(p) = ax + S(a)S(x) + S^2(a)S^2(x) + ... + S^p(a)S^p(x)$$

we can write

$$\begin{aligned}\sum(n) = \sum_{i=0}^{n} &(S^i(a) - S^{i+1}(a))(x + S(x) + ... + S^i(x)) + \\ &+ S^{n+1}(a)(x + S(x) + ... + S^n(x)).\end{aligned}$$

Consequently, for $n > m \geq 0$ one has

$$\begin{aligned}\sum(n) - \sum(m) &= \sum_{i=m+1}^{n} (S^i(a) - S^{i+1}(a))(x + S(x) + ... + S^i(x)) + \\ &+ S^{n+1}(a)(x + S(x) + ... + S^n(x)) - \\ &- S^{m+1}(a)(x + S(x) + ... + S^m(x)) = \\ &= \sum_{i=m+1}^{n} (S^i(a) - S^{i+1}(a))x^i + S^{n+1}(a)x^n - S^{m+1}(a)x^m.\end{aligned}$$

It follows that

$$\begin{aligned}\left|\sum(n) - \sum(m)\right| &\leq \left|S^{n+1}(a)x^n\right| + \left|S^{m+1}(a)x^m\right| + \\ &+ \sum_{i=m+1}^{n} \left|(S^i(a) - S^{i+1}(a))x^i\right| \leq \\ &\leq \left\|S^{n+1}(a)\right\| \left|x^n\right| + \left\|S^{m+1}(a)\right\| \left|x^m\right| + \\ &+ \sum_{i=m+1}^{n} \left\|x^i\right\| \left|S^i(a) - S^{i+1}(a)\right| .\end{aligned} \tag{2.8}$$

Due to the fact that $x \in C(S)$, we can find a number $M > 0$ such that

$$\|x^p\| < M, \tag{2.9}$$

for any natural p.

From (2.8) we obtain

$$\begin{aligned} \left|\sum(n) - \sum(m)\right| \leq \left\|S^{n+1}(a)\right\| |x^n| + \left\|S^{m+1}(a)\right\| |x^m| + \\ +M \sum_{i=m+1}^{n} \left|S^i(a) - S^{i+1}(a)\right|. \end{aligned} \tag{2.10}$$

But $a \geq S(a)$ implies $S^i(a) \geq S^{i+1}(a)$ and (2.10) becomes

$$\begin{aligned} \left|\sum(n) - \sum(m)\right| \leq \left\|S^{n+1}(a)\right\| |x^n| + \left\|S^{m+1}(a)\right\| |x^m| + \\ +M \sum_{i=m+1}^{n} (S^i(a) - S^{i+1}(a)) = \\ = M(S^{m+1}(a) - S^{n+1}(a)) + \left\|S^{n+1}(a)\right\| |x^n| + \left\|S^{m+1}(a)\right\| |x^m| \leq \\ \leq M(\left|S^{m+1}(a)\right| + \left|S^{n+1}(a)\right|) + \left\|S^{n+1}(a)\right\| |x^n| + \left\|S^{m+1}(a)\right\| |x^m|. \end{aligned}$$

Taking the norms we get:

$$\begin{aligned} \left\|\sum(n) - \sum(m)\right\| \leq M(\left\|S^{m+1}(a)\right\| + \left\|S^{n+1}(a)\right\|) + \\ + \left\|S^{n+1}(a)\right\| \|x^n\| + \left\|S^{m+1}(a)\right\| \|x^m\|. \end{aligned}$$

Using (2.9), we get

$$\left\|\sum(n) - \sum(m)\right\| \leq 2M(\left\|S^{m+1}(a)\right\| + \left\|S^{n+1}(a)\right\|), \tag{2.11}$$

for any $n > m \geq 0$.

Because of the assumption A3, for the already taken $\varepsilon > 0$ we can find a natural $p(\varepsilon)$ such that, for any $p \geq p(\varepsilon)$ one has

$$\left\|S^{p+1}(a)\right\| < \frac{\varepsilon}{4M}.$$

Consequently, for natural $n > m \geq p(\varepsilon)$, we obtain from (2.11) exactly $(\star)$. □

The following result is also concerned with the multiplicative structure of X.

Theorem 2.7. *Assume A1 is fulfilled. Then $AC(S)$ is a bilateral ideal in X.*

Proof. We have already seen that $AC(S)$ is a subspace of X. It remains to be proved that, for any $a \in AC(S)$ and any $x \in X$, one has $ax \in AC(S)$ and $xa \in AC(S)$. We shall do the proof for ax, the proof for xa being similar.

For any $n > m \geq 0$ one has

$$\left|S^m(|ax|) + S^{m+1}(|ax|) + ... + S^n(|ax|)\right| =$$

$$= S^m(|ax|) + S^{m+1}(|ax|) + ... + S^n(|ax|) \leq$$
$$\leq S^m(\|x\| \, |a|) + S^{m+1}(\|x\| \, |a|) + ... + S^n(\|x\| \, |a|) =$$
$$= \|x\| \left(S^m(|a|) + S^{m+1}(|a|) + ... + S^n(|a|) \right). \tag{2.12}$$

Because the series

$$\sum_{n=0}^{\infty} S^n(|a|)$$

converges, we find $p(\varepsilon)$ such that, for any $n > m \geq p(\varepsilon)$ one has

$$\|x\| \left\| S^m(|a|) + S^{m+1}(|a|) + ... + S^n(|a|) \right\| < \varepsilon.$$

Let us take $n > m \geq p(\varepsilon)$. Using (2.12), we get

$$\left\| S^m(|ax|) + S^{m+1}(|ax|) + ... + S^n(|ax|) \right\| \leq$$
$$\leq \|x\| \left\| S^m(|a|) + S^{m+1}(|a|) + ... + S^n(|a|) \right\| < \varepsilon$$

and this proves that the series

$$\sum_{n=0}^{\infty} S^n(|ax|)$$

converges a.s.o. □

3. Examples

Example 3.1. (Seminal Example) Actually, we began the preceding paragraph with this example, which motivates the present paper.

We take $X = c_0$ and $S : X \to X$ is the shift operator, given via

$$S(x) = y,$$

where, if

$$x = (x_0, x_1, ..., x_n, ..)$$

one has

$$y = (x_1, x_2, ..., x_n, ..).$$

It is easy to see that S is linear, continuous and $\|S\|_o = 1$.

Rephrasing the considerations from the beginning of the preceding paragraph, we get

$$C(S) = cc_0 = \text{the convergent series.}$$

At the same time (working for the real case), we have

$$AC(S) = l^1 = \text{the absolutely convergent series.}$$

We have already seen that (in the real case) the space $X = c_0$ and the shift operator S fulfill assumptions A1, A2 and A3.

a) We prove that $C(S) = cc_0$ is dense in $X = c_0$ (hence $C(S)$ is not closed). The same proof shows (in case $K = \mathbb{R}$) that $AC(S) = l^1$ is dense in c_0 (hence $AC(S)$ is not closed).

To this end, we pick an arbitrary $x = (x_n)_n \in c_0$. The sequence $(y(n))_n$ will be constructed as follows:

$$y(0) = (x_0, 0, 0, ..., 0, ..)$$

$$y(1) = (x_0, x_1, 0, ..., 0, ..)$$

$$...$$

$$y(n) = (x_0, x_1, ..., x_n, 0, 0, ..., 0, ...).$$

Then $y(n) \in cc_0$ and (in case $K = \mathbb{R}$) $y(n) \in l^1$.

On the other hand, one can see that, for any n:

$$\|y(n) - x\| = \sup_{p>n} |x_p| \underset{n}{\to} 0,$$

thus finishing the proof.

b) We shall compute the values of $T_S : cc_0 \to c_0$.

Let us take an arbitrary element

$$x = (x_0, x_1, ..., x_n, ..) \in c_0$$

and write

$$t(x) = \sum_{n=0}^{\infty} x_n$$

and

$$\sigma_n = x_0 + x_1 + ... + x_n,$$

for any $n \in \mathbb{N}$.

We shall prove that

$$T_S(x) \stackrel{def}{=} x^\infty = (t(x), t(x) - x_0, t(x) - (x_0 + x_1), ..., t(x) - \sigma_{n-1}, ..).$$

Because

$$T_S(x) = \lim_p x^p$$

it remains to be proved that

$$\lim_p \|x^p - x^\infty\| = 0.$$

The n-th component of x^p is equal to

$$x_n + x_{n+1} + ... + x_{n+p}$$

and the n-th component of x^∞ is equal to

$$t(x) - \sigma_{n-1},$$

hence the n-th component of $x^p - x^\infty$ is equal to

$$x_n + x_{n+1} + ... + x_{n+p} - t(x) + \sigma_{n-1} = \sigma_{n+p} - t(x).$$

It follows that, for any p one has

$$\|x^p - x^\infty\| = \sup_n |\sigma_{n+p} - t(x)|.$$

Taking an arbitrary $\varepsilon > 0$, we can find $p(\varepsilon) \in \mathbb{N}$ such that, for any $q \geq p(\varepsilon)$ one has

$$|\sigma_q - t(x)| < \varepsilon$$

and this implies

$$\|x^p - x^\infty\| \leq \varepsilon,$$

for any $p \geq p(\varepsilon)$.

c) It is obvious that, for any $x \in X = c_0$, one has

$$\lim_n S^n(x) = 0.$$

Consequently, Theorem 2.3 implies the fact that $(I - S)_1 : X \to C(S)$ is bijective. Hence $I - S : c_0 \to c_0$is injective and $(I - S)(c_0) = cc_0$ which is dense in c_0. Therefore 1 is in the continuous spectrum of S and T_S is discontinuous (Corollary 2.1).

d) The discontinuity of T_S can be proved in another way.

Namely, we shall prove that the functional $L : cc_0 \to K$, given via

$$L(x) = t(x)$$

is discontinuous.

Accepting this fact, the discontinuity of T_S follows from the relation

$$L = \pi_0 \circ T_S,$$

where $\pi_0 : c_0 \to K$ is the (linear and continuous) projection number zero given via

$$\pi_0((x_n)_n) = x_0.$$

Now, let us turn to the proof of the fact that L is discontinuous. We must prove the existence of an $\varepsilon_0 > 0$ having the property that, for any $\delta > 0$ one can find $x \in cc_0$ with $\|x\| \leq \delta$ and yet $|L(x)| > \varepsilon_0$.

To this end we can begin with an arbitrary $\varepsilon_0 > 0$. Let also $\delta > 0$ be arbitrarily taken. We consider $y = (y_n)_n$ a semiconvergent series with real terms, e.g.

$$y = \left(\frac{(-1)^n}{n+1}\right)_{n \geq 0}.$$

One can find $n(\delta) \in \mathbb{N}$ such that, for any $n \geq n(\delta)$ one has $|y_n| < \delta$. We construct the semiconvergent series $z = (z_n)_{n \geq 0}$, where $z_0 = y_{n(\delta)}$, $z_1 = y_{n(\delta)+1}$, ..., $z_n = y_{n(\delta)+n}$, ... and, of course, $\|z\| \leq \delta$.

Using Riemann's permutation theorem, we can find a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the series

$$x = (z_{\pi(0)}, z_{\pi(1)}, ..., z_{\pi(n)}, ...)$$

has the sum equal to $\varepsilon_0 + 1$. Hence $\|x\| \leq \delta$ and $|L(x)| = L(x) = \varepsilon_0 + 1 > \varepsilon_0$.

e) *Because $T_S : cc_0 \to c_0$ is a bijection, as we have seen, we have on cc_0 two norms.*

The first norm on cc_0 is the norm induced by the norm of c_0, having the analytical expression

$$\|x\| = \sup_n |x_n|,$$

where

$$x = (x_0, x_1, ..., x_n, ...).$$

With respect to this norm, cc_0 is not a Banach space (because cc_0 is not closed in c_0).

The second norm on cc_0 is obtained via T_S-transport from the norm of c_0. For $x \in cc_0$, this norm will be

$$|||x||| \stackrel{def}{=} \|T_S(x)\|.$$

Hence, the analytical expression of this norm will be

$$|||x||| = \sup\{|t(x)|, |t(x) - \sigma_0|, |t(x) - \sigma_1|, ..., |t(x) - \sigma_n|, ...\}$$

with previous notations. With respect to this norm, cc_0 is a Banach space.

f) *Now we shall see that Theorem* 2.5 *generalizes D'Alembert's criterion.*

Namely, we consider a "series" $x = (x_n)_n \in cc_0$ and assume that the hypothesis of Theorem 2.5 is fulfilled.

We have, for any natural n:

$$S^{n+1}(x) = (x_{n+1}, x_{n+2}, ...)$$

and

$$S^n(x) = (x_n, x_{n+1}, ...).$$

Hence, the hypothesis says that there exist $0 < a < 1$ and $M \in N$ such that

$$A_{n+1} = \sup_{p \geq n+1} |x_p| \leq a \sup_{p \geq n} |x_p| = aA_n,$$

for all $n \geq M$.

In case $A_{n+1} = A_n$ it follows that $A_n = A_{n+1} = 0$ for all $n \geq M$, hence $x_n = 0$ for all $n \geq M$ and the series

$$\sum_{n=0}^{\infty} x_n \tag{3.1}$$

converges (trivially).

So, the case when some $A_n = 0$ is trivial and let us see what happens when $A_n > 0$ for all n.

It follows that, for all $n \geq M$, one has

$$0 < A_{n+1} \leq aA_n,$$

hence

$$0 < A_{n+1} < A_n,$$

and, consequently

$$A_n = \sup_{p \geq n} |x_p| = \sup_{p=n} |x_p| = |x_n| \,.$$

Then, for all $n \geq M$, one has

$$\sup_{p \geq n+1} |x_p| \leq a \, |x_n|$$

which implies

$$|x_{n+1}| \leq a \, |x_n| \tag{3.2}$$

and the series (3.1) converges.

In case all the x_n are non null, we can write (3.2) in the form: for all $n \geq M$

$$\left| \frac{x_{n+1}}{x_n} \right| \leq a \tag{3.3}$$

and this implies (of course)

$$\limsup_n \left| \frac{x_{n+1}}{x_n} \right| \leq a < 1. \tag{3.4}$$

It is readily seen that, in case (3.4) is fulfilled, one can find a natural M, such that, for all $n \geq M$, one has (3.3). This completes the proof of the fact that Theorem 2.5 is exactly D'Alembert's criterion in this special case.

g) At this point we begin with the remark that Theorem 2.7 is applicable. *Hence* l^1 *is a bilateral ideal in* c_0.

Finally, we shall be concerned with Theorem 2.6 in this particular case. Interpreting the result we obtain the following

FACT. *Assume* $a = (a_n)_n$ *is a decreasing sequence which converges to zero. Let* $(x_n)_n$ *be a sequence such that the series*

$$\sum_{n=0}^{\infty} x_n$$

is convergent.

Then, the series

$$\sum_{n=0}^{\infty} a_n x_n$$

is convergent.

This Fact is equivalent to the apparently more general following result:

Abel-Dirichlet Criterion *Let* $(b_n)_n$ *be a decreasing sequence which is bounded. Let* $(x_n)_n$ *be a sequence such that the series*

$$\sum_{n=0}^{\infty} x_n$$

is convergent.

Then, the series

$$\sum_{n=0}^{\infty} b_n x_n$$

is convergent.

Indeed, assuming the validity of the Fact, let us write, for $b = \lim_n b_n$:

$$b_n = a_n + b,$$

where $a_n = b_n - b$.

Then $(a_n)_n$ is decreasing and converges to 0.

It follows that the series

$$\sum_{n=0}^{\infty} a_n x_n$$

converges.

Because the series

$$\sum_{n=0}^{\infty} b x_n$$

converges too, it follows that the sum of the preceding series, i.e. the series

$$\sum_{n=0}^{\infty} b_n x_n$$

is convergent.

We proved that Theorem 2.6 generalizes the Abel-Dirichlet Criterion.

Example 3.2. We consider an arbitrary Banach space X. Let X_1 and X_2 be two closed subspaces of X which are complementary (i.e. each element $x \in X$ can be written uniquely in the form

$$x = x_1 + x_2,$$

with $x_1 \in X_1$ and $x_2 \in X_2$).

We consider the linear and continuous projections $P_i : X \to X$ associated with X_i, $i = 1, 2$ (i.e. P_i is given via

$$P_i(x) = x_i,$$

for $i \in \{1, 2\}$).

Let us take $S = P_1$. Then $I - S = P_2$. Because $S^n = S$ for any natural $n \geq 1$, we use Theorem 2.3 and get

$$C(S) = (I - S)(X) = P_2(X) = X_2.$$

For any $x \in X_2 = C(S)$, one has

$$S^n(x) = 0,$$

for any natural $n \geq 1$, hence

$$T_S(x) = \sum_{n=0}^{\infty} S^n(x) = x = x - \lim_n S^n(x).$$

Example 3.3. Let us consider the real numbers $a < b$. The Banach space will be $X = C[a, b]$. The operator $S : X \to X$ will be defined as follows:

$$S(f) = g,$$

where $g : [a, b] \to K$ is given via

$$g(x) = \int_a^x f(t)dt.$$

It is seen that S is linear, continuous and $\|S\|_o = b - a$.

The last assertions follow from the inequality

$$\left|\int_a^x f(t)dt\right| \leq \int_a^b |f(t)|\, dt \leq (b-a)\,\|f\|\,,$$

which is valid for any $x \in [a,b]$, and from the fact that

$$\|S(u)\| = b-a,$$

where u is the constant function equal to 1.

It is known that, for any $f \in X$ and any natural $n \geq 1$, one has

$$S^n(f) = g,$$

where $g : [a,b] \to K$ is given via

$$g(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt.$$

a) *We claim that one has (see also Corollary* 2.1, 1.*)*

$$C(S) = X.$$

Indeed, for any $f \in X$, any $n \geq 1$ and any $x \in [a,b]$, we have successively

$$|S^n(f)(x)| = \left|\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt\right| \leq \int_a^b \frac{|x-t|^{n-1}}{(n-1)!}\,\|f\|\,dt \leq$$

$$\leq \int_a^b \frac{|b-a|^{n-1}}{(n-1)!}\,\|f\|\,dt = \frac{|b-a|^n}{(n-1)!}\,\|f\| = a_n.$$

Because the series

$$\sum_{n=1}^{\infty} a_n$$

is convergent (with sum $(b-a)e^{b-a}\,\|f\|$) it follows that the series of functions

$$\sum_{n=0}^{\infty} S^n(f)$$

converges uniformly and absolutely, hence converges in X.

Working for $K = \mathbb{R}$, it is seen that

$$AC(S) = X.$$

As a consequence (see Theorem 2.3), we have the linear and continuous operators $I - S : X \to X$ and $T_S : X \to X$ and

$$T_S = (I - S)^{-1}.$$

b) *It is possible to express T_S more concretely.*

To this end, we take an arbitrary $f \in X$ and an arbitrary $x \in [a, b]$. Due to the uniform convergence, we get

$$T_S(f)(x) = f(x) + \sum_{n=1}^{\infty} S^n(f)(x) =$$

$$= f(x) + \sum_{n=1}^{\infty} \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt = f(x) + \int_a^x (\sum_{n=0}^{\infty} \frac{(x-t)^n}{n!}) f(t)dt =$$

$$= f(x) + \int_a^x e^{x-t} f(t)dt.$$

We used the fact that, for fixed $x \in [a, b]$, the series of functions

$$\sum_{n=1}^{\infty} \varphi_{x,n}$$

converges uniformly on $[a, x]$, where

$$\varphi_{x,n}(t) = \frac{(x-t)^{n-1}}{(n-1)!} f(t)$$

$\left(\text{because } \left|\varphi_{x,n}(t)\right| \leq \frac{(x-a)^{n-1}}{(n-1)!} \|f\|\right)$.

Taking into account the previous facts, we proved that

$$T_S = (I - S)^{-1} = I + V.$$

Here $S : X \to X$ is the linear and continuous integral operator with kernel $Q : [a, b]^2 \to K$ given via

$$Q(x, t) = \begin{cases} 0, & \text{if } x < t \\ 1, & \text{if } x \geq t \end{cases}$$

(it is clear that, for any $x \in [a, b]$, one has

$$S(f)(x) = \int_a^b Q(x, t) f(t)dt).$$

Also $V : X \to X$ is the linear and continuous integral operator with kernel $P : [a,b]^2 \to K$ given via

$$P(x,t) = \begin{cases} 0, & \text{if } x < t \\ e^{x-t}, & \text{if } x \geq t \end{cases}$$

(namely, for any $x \in [a,b]$, one has

$$V(f)(x) = \int_a^b P(x,t)f(t)dt).$$

Remark 3.1. One can obtain $T_S = (I - S)^{-1}$ in another way.

Let us give an arbitrary $f \in X = C(S)$. We want to obtain $g \in X$ such that $g = T_S(f)$, i.e. $f = (I - S)(g)$. Writing $y : [a,b] \to K$,

$$y(x) = \int_a^x g(t)dt$$

(i.e. $y = S(g)$) it is seen that y is the solution of the Cauchy problem

$$y' = g \text{ and } y(a) = 0.$$

The relation $f = (I - S)(g)$ becomes the differential equation (Cauchy problem)

$$y' - y = f$$

with the initial condition $y(a) = 0$.

The solution of the homogeneous equation

$$y' - y = 0$$

is of the form

$$y(x) = Ce^x.$$

Using the variation of constants method, we write

$$y(x) = C(x)e^x,$$

which implies

$$y'(x) - y(x) = C'(x)e^x = f(x).$$

Hence

$$C(x) = \int_a^x f(t)e^{-t}dt + A.$$

Because $y(a) = 0$, one must have $C(a) = 0$, consequently $A = 0$.
So

$$y(x) = e^x \int_a^x f(t)e^{-t}dt,$$

for any $x \in [a, b]$, which implies

$$g(x) = y^{'}(x) = e^x \int_a^x f(t)e^{-t}dt + e^x f(x)e^{-x} =$$

$$= f(x) + \int_a^x f(t)e^{x-t}dt.$$

We got again

$$g(x) = (I + V)(f)(x).$$

c) *In view of the existence of the bijections* $I - S : X \to X$ *and* $I + V = (I - S)^{-1} : X \to X$ *we obtain via transport two new Banach spaces norms on* X.

These norms are given as follows:

$$|||f||| \stackrel{def}{=} \|(I - S)(f)\| = \sup_{x\in[a,b]} \left| f(x) - \int_a^x f(t)dt \right|$$

and

$$||||f|||| \stackrel{def}{=} \|(I + V)(f)\| = \sup_{x\in[a,b]} \left| f(x) + \int_a^x e^{x-t} f(t)dt \right|.$$

It is seen that, for any $x \in [a, b]$ and any $f \in X$, we have:

$$\left| f(x) - \int_a^x f(t)dt \right| \le |f(x)| + \int_a^x |f(t)|\, dt \le \|f\| + (b - a)\, \|f\|$$

and

$$\left| f(x) + \int_a^x e^{x-t} f(t)dt \right| \le |f(x)| + \int_a^x e^{b-a}\, |f(t)|\, dt \le \|f\| + e^{b-a}(b - a)\, \|f\| ,$$

hence

$$|||f||| \le (1 + (b - a))\, \|f\|$$

and

$$||||f|||| \le (1 + e^{b-a}(b - a))\, \|f\| .$$

It follows that *the norms* $\|\|$, $|||\,|||$ *and* $\|\|\|\|\|\|$ *are equivalent.*

Problem. Find (as small as possible) constants $A > 0$ and $B > 0$ such that

$$|||f||| \le A \|f\|$$

and

$$\|\|\|f\|\|\| \le B \|f\|,$$

for any $f \in X$.

Remark 3.2. Working for $K = \mathbb{C}$, we know that the spectral radius of S is equal to 0, hence the spectrum of S reduces to $\{0\}$ = the residual spectrum of S and (of course) 1 is in the resolvent set of S.

Example 3.4. We shall consider the space $X = L^p(\mu)$ where $1 \le p < \infty$ and μ is the Lebesgue measure on $[0,1]$. The linear and continuous operator $S : L^p(\mu) \to L^p(\mu)$ will be given via

$$S(\widetilde{f}) = \widetilde{g},$$

where

$$g(t) = tf(t)$$

for all $t \in [0,1]$.

It is seen that (in case $K = \mathbb{R}$) the operator S is positive.

For any natural n, one has

$$S^n(\widetilde{f}) = \widetilde{g},$$

where

$$g(t) = t^n f(t),$$

for all $t \in [0,1]$.

a) *One can prove that*

$$\lim_n S^n(\widetilde{f}) = 0,$$

for any $\widetilde{f} \in X$.

Indeed, let $\widetilde{f} \in L^p(\mu) = X$ and take a representative $f \in \widetilde{f}$. Then, for any $t \in [0,1)$, one has

$$\lim_n t^n f(t) = 0,$$

hence

$$\lim_n t^n f(t) = 0 \ \mu\text{-a.e. and } |t^n f(t)| \le |f(t)|,$$

for all $t \in [0,1]$.

Using Lebesgue's dominated convergence theorem (see [5]) we get

$$\widetilde{f_n} \underset{n}{\to} 0$$

in $L^p(\mu)$, where $f_n(t) = t^n f(t)$ for all $t \in [0,1]$.

Hence

$$S^n(\widetilde{f}) \underset{n}{\to} 0.$$

b) *Using the result from a), we can prove that the following assertions are equivalent for a measurable function* $f : [0,1] \to K$*:*

1°. $\widetilde{f} \in C(S)$.

2°. *There exists* $u \in \mathcal{L}^p(\mu)$ *such that*

$$f(t) = (1-t)u(t) \ \mu\text{- a.e. }.$$

3°. *One has* $g \in \mathcal{L}^p(\mu)$*, where*

$$g(t) = \begin{cases} \frac{1}{1-t} f(t), & \text{if } 0 \le t < 1 \\ 0, & \text{if } t = 1 \end{cases}.$$

4°. $\widetilde{f} \in AC(S)$ *(for* $K = \mathbb{R}$*).*

The equivalence $1^\circ \iff 2^\circ$ follows from Theorem 2.3, points 3. and 4. and from a): we have

$$C(S) = (I-S)(X)$$

and the elements $\widetilde{f} \in (I-S)(X)$ have representatives given by

$$f(t) = (1-t)u(t),$$

with $u \in \mathcal{L}^p(\mu)$.

$2^\circ \Rightarrow 3^\circ$ is obvious.

The implication $3^\circ \Rightarrow 1^\circ$ (which will be proved immediately) establishes the equivalence of assertions $1^\circ, 2^\circ, 3^\circ$.

We consider the sequence $(f_n)_n$ in $\mathcal{L}^p(\mu)$, given via

$$f_n(t) = \begin{cases} (1 + t + t^2 + ... + t^n) f(t), & \text{if } 0 \le t < 1 \\ 0, & \text{if } t = 1 \end{cases}.$$

It is seen that for any n and t one has

$$|f_n(t)| \le |g(t)|$$

and

$$f_n(t) \underset{n}{\to} g(t) \ \mu\text{- a.e. }.$$

Again Lebesgue's dominated convergence theorem implies that

$$f_n \underset{n}{\to} g$$

in $\mathcal{L}^p(\mu)$.

But

$$\widetilde{f_n} = \widetilde{f} + S(\widetilde{f}) + ... + S^n(\widetilde{f})$$

and we obtain the fact that

$$\widetilde{f_n} \underset{n}{\to} \widetilde{g}$$

in $L^p(\mu)$, which can be rephrased as follows: the series

$$\sum_{n=1}^{\infty} S^n(\widetilde{f})$$

converges to $\widetilde{g}$ in $L^p(\mu)$.

Hence

$$\widetilde{f} \in C(S).$$

In order to finish the proof we must prove that $1° \Rightarrow 4°$.

So, let us take $\widetilde{f} \in C(S)$.

Because $1° \Rightarrow 3°$, we obtain (taking a representative $f \in \widetilde{f}$) the function $g \in \mathcal{L}^p(\mu)$ constructed with the aid of f as in the statement of $3°$.

It follows that $|g| \in \mathcal{L}^p(\mu)$, hence the function $u : [0,1] \to K$ given via

$$u(t) = \begin{cases} \frac{1}{1-t} \left| f \right| (t), & \text{if } 0 \le t < 1 \\ 0, & \text{if } t = 1 \end{cases}$$

is in $\mathcal{L}^p(\mu)$.

From $3° \Rightarrow 1°$ we get $\left|\widetilde{f}\right| \in C(S)$, i.e. $\widetilde{f} \in AC(S)$.

c) *We consider the subspace $H \subset X$ which consists of "constant functions". More precisely*

$$H = \{\widetilde{f} \in X \mid f \text{ is constant}\}.$$

We shall prove that $H \cap C(S) = \{0\}$ (and this shows that $C(S) \neq X$).

Indeed, assume by absurd, the existence of $0 \neq \widetilde{f} \in H \cap C(S)$. Let $0 \neq \alpha \in K$ be such that $f(t) = \alpha$ μ- a.e. . Using $3°$, from *b*) we obtain the function $g \in \mathcal{L}^p(\mu)$ given as follows

$$g(t) = \begin{cases} \frac{\alpha}{1-t}, & \text{if } 0 \le t < 1 \\ 0, & \text{if } t = 1 \end{cases}.$$

A simple computation shows that

$$\int |g|^p \, d\mu = \infty$$

and we got a contradiction.

d) *Now we consider the subspace*

$$Y = \{\widetilde{f} \in L^p(\mu) \mid \textit{there exists } 0 < a < 1 \textit{ such that } f(t) = 0 \textit{ for all } t \in [a, 1]\}.$$

(of course, a changes with f).

We shall prove that Y is dense in X.

To this end, we take first an element $\widetilde{u} \in L^p(\mu)$ such that $u(t) \geq 0$ μ-a.e. . We define the sequence $(u_n)_n$ in $\mathcal{L}^p(\mu)$ such that

$$u_n(t) = \begin{cases} u(t), & \text{if } 0 \leq t \leq 1 - \frac{1}{n} \\ 0, & \text{if } 1 - \frac{1}{n} < t \leq 1 \end{cases},$$

for $n \geq 2$.

Then $\widetilde{u_n} \in Y$ and

$$|u_n| = u_n \leq u,$$

for all n.

We have also

$$\lim_n u_n(t) = u(t) \ \mu\text{-a.e. } .$$

Lebesgue's dominated convergence theorem says that

$$\widetilde{u_n} \underset{n}{\to} \widetilde{u}$$

in $L^p(\mu) = X$.

For an arbitrary real $f : [0,1] \to \mathbb{R}$, $\widetilde{f} \in X$, we write $f = u - v$ with $u, v : [0,1] \to \mathbb{R}$, u and v being positive functions in $\mathcal{L}^p(\mu)$.

We consider the sequences $(\widetilde{u_n})_n$ and $(\widetilde{v_n})_n$ in Y such that $\widetilde{u_n} \underset{n}{\to} \widetilde{u}$ and $\widetilde{v_n} \underset{n}{\to} \widetilde{v}$ in X and

$$\widetilde{u_n} - \widetilde{v_n} \underset{n}{\to} \widetilde{f},$$

where $\widetilde{u_n} - \widetilde{v_n} \in Y$.

For complex (in case $K = \mathbb{C}$) $f : [0,1] \to \mathbb{C}$, $\widetilde{f} \in X$, we write $\widetilde{f} = \widetilde{g} + i\widetilde{h}$ and find real $\widetilde{g_n}, \widetilde{h_n}$ in Y such that $\widetilde{g_n} \underset{n}{\to} \widetilde{g}$ and $\widetilde{h_n} \underset{n}{\to} \widetilde{h}$ in Y. Then $\widetilde{g_n} + i\widetilde{h_n} \in Y$ and $\widetilde{g_n} + i\widetilde{h_n} \underset{n}{\to} \widetilde{f}$.

e) *Now we prove that*

$$Y \subset C(S).$$

Let us take $\widetilde{f}$ in Y and prove that $\widetilde{f} \in C(S)$. According to b) 3°, we must prove that (taking a representative $f \in \widetilde{f}$) the function $g : [0,1] \to K$, given via

$$g(t) = \begin{cases} \frac{f(t)}{1-t}, & \text{if } 0 \le t < 1 \\ 0, & \text{if } t = 1 \end{cases}$$

is in $\mathcal{L}^p(\mu)$.

There exists $0 < a < 1$ such that $f(t) = 0$ for $a \le t \le 1$.

It follows that

$$g(t) = \begin{cases} \frac{f(t)}{1-t}, & \text{if } 0 \le t < a \\ 0, & \text{if } a \le t \le 1 \end{cases}.$$

For $0 \le t < a$, one has

$$0 < \frac{1}{1-t} < \frac{1}{1-a}.$$

Consequently

$$|g(t)| \le \frac{1}{1-a} |f(t)|,$$

for all $t \in [0,1]$ and $g \in \mathcal{L}^p(\mu)$.

f) *We conclude that*

$$C(S) = AC(S) = (I - S)(X).$$

According to point c, it follows that $C(S) \neq X$ and $C(S)$ is meager.

According to point d) and point e), it follows that $C(S)$ is dense in X.

Theorem 2.3 says that (using point a)) we have the mutually inverse bijections

$$(I - S)_1 : X \to C(S) \text{ and } T_S : C(S) \to X.$$

Hence 1 is in the continuous spectrum of S and T_S is discontinuous. Notice that $T_S : C(S) \to X$ acts as follows: $T_S(\widetilde{f}) = \widetilde{g}$ where

$$g(t) = \begin{cases} \frac{1}{1-t} f(t), & \text{if } 0 \le t < 1 \\ 0, & \text{if } t = 1 \end{cases}$$

for any representative $f \in \widetilde{f} \in C(S)$.

Remark 3.3. Actually, the spectrum of S is equal to $[0,1]$ and coincides with the continuous spectrum of S.

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