Smooth vectors and Weyl-Pedersen calculus for representations of nilpotent Lie groups

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Abstract - We present some recent results on smooth vectors for unitary irreducible representations of nilpotent Lie groups. Applications to the Weyl-Pedersen calculus of pseudo-differential operators with symbols on the coadjoint orbits are also discussed.

Key words and phrases : Weyl calculus, nilpotent Lie group, semidirect product.

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1. Introduction

'Weyl-Pedersen calculus' is the name proposed in [4] for the remarkable correspondence $a \mapsto \mathrm{Op}^\pi(a)$ constructed by N.V. Pedersen in [49] as a generalization of the pseudo-differential Weyl calculus on $\mathbb{R}^n$. Here $\pi\colon G \to \mathcal{B}(\mathcal{H})$ is any unitary irreducible representation of a connected, simply connected, nilpotent Lie group G, the symbol a can be any tempered distribution on the coadjoint orbit $\mathcal{O}$ corresponding to π by the orbit method of [35], and $\mathrm{Op}^\pi(a)$ is a linear operator in the representation space $\mathcal{H}$, which is in general unbounded. When π is the Schrödinger representation of the $(2n+1)$-dimensional Heisenberg group, the correspondence $a \mapsto \mathrm{Op}^\pi(a)$ is precisely the calculus suggested by H. Weyl in [56] for applications to quantum mechanics. This calculus was later developed by L. Hörmander in [27] and made into a powerful calculus of pseudo-differential operators on $\mathbb{R}^n$; see [28].

In the present paper we discuss the classical notion of smooth vectors —and the related notion of smooth operators— with a view toward their

crucial importance for the Weyl-Pedersen calculus. We then approach a
related circle of ideas that recently emerged in [4], namely the modulation
spaces for unitary irreducible representations of nilpotent Lie groups. We
take the opportunity of this discussion to extend some known facts to a set-
ting where they hold true in a natural degree of generality (see for instance
Theorem 3.1 below). We also take a close look at some new examples of
unitary irreducible representations and find out their related notions which
illustrate the main theme of the present paper: the preduals of the corre-
sponding coadjoint orbits, their ambiguity function, or their space of smooth
vectors (see Proposition 5.1 and Corollary 5.1).

Let us mention that the importance of the Weyl-Pedersen calculus and
the related circle of ideas goes far beyond the framework of representation
theory of nilpotent Lie groups. Many other interesting developments within
the theory of partial differential equations and the finite-dimensional Lie the-
ory can be found for instance in the references [1], [2], [30], [44], [43], [31],
[32], [25], [45], [17], [39], [40], and [38]. Moreover, one can use a similar con-
struction even for representations of certain *infinite-dimensional Lie groups*
in order to provide a geometric explanation for the gauge covariance for the
magnetic Weyl calculus of [41], [33], [34], [42] and the references therein.
The representation theoretic approach to the magnetic Weyl calculus has
been taken up in the papers [3], [6]; see also the survey [5].

The structure of the present paper is summarized in the following table
of contents:

Contents

Notation and background. Throughout the paper we denote by $\mathcal{S}(\mathcal{V})$ the
Schwartz space on a finite-dimensional real vector space $\mathcal{V}$. That is, $\mathcal{S}(\mathcal{V})$
is the set of all smooth functions that decay faster than any polynomial

together with their partial derivatives of arbitrary order. Its topological dual —the space of tempered distributions on $\mathcal{V}$— is denoted by $\mathcal{S}'(\mathcal{V})$. We shall also have the occasion to use these notions with $\mathcal{V}$ replaced by a coadjoint orbit of a nilpotent Lie group. In this situation we need the notion of polynomial structure on a manifold; see Sect. 1 in [48] for details. We use $\langle \cdot, \cdot \rangle$ to denote any duality pairing between finite-dimensional real vector spaces whose meaning is clear from the context.

We shall also use the convention that the Lie groups are denoted by upper case Latin letters and the Lie algebras are denoted by the corresponding lower case Gothic letters.

As regards the background information for the present paper, we refer to [28], [18], and [22] for basic notions of pseudo-differential Weyl calculus on $\mathbb{R}^n$. The necessary notions of representation theory for nilpotent Lie groups (in particular, the correspondence between the coadjoint orbits and the unitary irreducible representations) can be found in [50], [11], and [37]; see also [55] and [36]. Our references for topological vector spaces, nuclear spaces, and related topics are [53], [54], and [9].

2. Smooth vectors for Lie group representations

The smooth vectors have been a basic tool in representation theory of Lie groups; see for instance the early paper [19] and the classical monographs [55] and [36]. In this section we discuss some of the very basic properties of the smooth vectors for the purpose of providing the necessary background for the later developments in the present paper.

Notation 2.1. Throughout this section we shall use the following notation:

- G is a connected unimodular Lie group with the Lie algebra $\mathfrak{g}$;

- $\mathrm{d}x$ denotes a fixed Haar measure on G;

- $\mathcal{V}$ and $\mathcal{Y}$ are some complex Banach spaces;

- $\pi \colon G \to \mathcal{B}(\mathcal{Y})$ is a representation which is continuous, in the sense that for every $x \in \mathcal{Y}$ the mapping $\pi(\cdot)x \colon G \to \mathcal{Y}$ is continuous.

2.1. Distribution theory on Lie groups

Some references for distribution theory on Lie groups are [7] and [55]. The present subsection just records a few basic notions and properties needed later.

Definition 2.1. *We define the **spaces of test functions** on the Lie group G as follows:*

1. The space
$$\mathcal{E}(G, \mathcal{V}) := \{\phi \colon G \to \mathcal{V} \mid \phi \text{ is smooth}\}$$

with the usual topology of a Fréchet space (given by the uniform convergence on compact sets of functions and their partial derivatives in local charts).

2. The space
$$\mathcal{D}(G, \mathcal{V}) := \{\phi \in \mathcal{E}(G, \mathcal{V}) \mid \operatorname{supp} \phi \text{ is compact}\}$$

with the usual topology of an inductive limit of Fréchet spaces.

If $\mathcal{V} = \mathbb{C}$ then we denote simply $\mathcal{E}(G, \mathbb{C}) = \mathcal{E}(G)$ and $\mathcal{D}(G, \mathbb{C}) = \mathcal{D}(G)$. For every integer $m \geq 1$ we shall also need the function space

$$\mathcal{C}_0^m(G) := \{\phi \colon G \to \mathbb{C} \mid \phi \text{ is of class } \mathcal{C}^m \text{ and } \operatorname{supp} \phi \text{ is compact}\}$$

with its usual topology of an inductive limit of Banach spaces.

We then define the **spaces of vector valued distributions**

$$\mathcal{D}'^{\mathcal{Y}}(G, \mathcal{V}) := \{u \colon \mathcal{D}(G, \mathcal{V}) \to \mathcal{Y} \mid u \text{ is linear and continuous}\}$$

and

$$\mathcal{E}'^{\mathcal{Y}}(G, \mathcal{V}) := \{u \colon \mathcal{E}(G, \mathcal{V}) \to \mathcal{Y} \mid u \text{ is linear and continuous}\}$$

and endow them with the topology of pointwise convergence. We denote the evaluation mapping by

$$\langle \cdot, \cdot \rangle \colon \mathcal{D}'^{\mathcal{Y}}(G, \mathcal{V}) \times \mathcal{D}(G, \mathcal{V}) \to \mathcal{Y}, \quad \langle u, \phi \rangle := u(\phi),$$

and similarly for $\langle \cdot, \cdot \rangle \colon \mathcal{E}'^{\mathcal{Y}}(G, \mathcal{V}) \times \mathcal{E}(G, \mathcal{V}) \to \mathcal{Y}$.

For $\mathcal{V} = \mathbb{C}$ we denote simply $\mathcal{E}'^{\mathcal{Y}}(G, \mathbb{C}) = \mathcal{E}'^{\mathcal{Y}}(G)$ and $\mathcal{D}'^{\mathcal{Y}}(G, \mathbb{C}) = \mathcal{D}'^{\mathcal{Y}}(G)$. If in addition we have $\mathcal{Y} = \mathbb{C}$, then we further denote $\mathcal{E}'^{\mathcal{Y}}(G) = \mathcal{E}'(G)$ and $\mathcal{D}'^{\mathcal{Y}}(G) = \mathcal{D}'(G)$.

Definition 2.2. The **support** of the distribution $u \in \mathcal{D}'^{\mathcal{Y}}(G, \mathcal{V})$ is denoted by $\operatorname{supp} u$ and is defined as the intersection of all the closed sets $F \subseteq G$ such that for every $\phi \in \mathcal{D}(G, \mathcal{V})$ with $F \cap \operatorname{supp} \phi = \emptyset$ we have $\langle u, \phi \rangle = 0$.

Remark 2.1. Let $L^1_{\operatorname{loc}}(G)$ denote the linear space of (equivalence classes of) measurable functions on G which are absolutely integrable with respect to the Haar measure $\mathrm{d}x$ on every compact subset of G. Then there exists a natural linear embedding $L^1_{\operatorname{loc}}(G) \hookrightarrow \mathcal{D}'(G)$. Specifically, every $f \in L^1_{\operatorname{loc}}(G)$ gives rise to a distribution also denoted by f and defined by

$$(\forall \phi \in \mathcal{D}(G)) \quad \langle f, \phi \rangle = \int_G f\phi \mathrm{d}x.$$

Note that $L^1_{\mathrm{loc}}(G)$ contains many function spaces on G, like the space of continuous functions, or the space of smooth functions $\mathcal{E}(G)$, or the Lebesgue space $L^p(G)$ if $1 \le p \le \infty$ etc.

Remark 2.2. We have

$$\mathcal{E}'(G) = \{u \in \mathcal{D}'(G) \mid \mathrm{supp}\, u \text{ is compact}\}.$$

For every compact set $K \subseteq G$ we denote $\mathcal{E}'_K(G) = \{u \in \mathcal{D}'(G) \mid \mathrm{supp}\, u \subseteq K\}$.

Remark 2.3. We recall from [53] and [54] that the locally convex spaces $\mathcal{E}(G)$ and $\mathcal{D}(G)$ are nuclear. Moreover, we have the linear topological isomorphisms

$$\mathcal{E}(G, \mathcal{Y}) \simeq \mathcal{E}(G) \widehat{\otimes} \mathcal{Y} \text{ and } \mathcal{D}(G, \mathcal{Y}) \simeq \mathcal{D}(G) \widehat{\otimes} \mathcal{Y},$$

which are natural in the sense that every pair $(\phi, y) \in \mathcal{E}(G) \times \mathcal{Y}$ corresponds to the function $\phi(\cdot)y \in \mathcal{E}(G, \mathcal{Y})$. Also recall the linear topological isomorphisms

$$\mathcal{E}(G) \widehat{\otimes} \mathcal{E}(G) \simeq \mathcal{E}(G \times G) \quad \text{and} \quad \mathcal{D}(G) \widehat{\otimes} \mathcal{D}(G) \simeq \mathcal{D}(G \times G)$$

that take a pair $(\phi_1, \phi_2) \in \mathcal{D}(G) \times \mathcal{D}(G)$ to the function $\phi_1 \otimes \phi_2$ defined by $(x_1, x_2) \mapsto \phi_1(x_1)\phi_2(x_2)$.

Example 2.1. Here are some examples of vector valued distributions that will be needed in the sequel.

1. For arbitrary $g \in G$ the $\mathcal{Y}$-valued *Dirac distribution* $\delta_g^{\mathcal{Y}} \in \mathcal{E}'^{\mathcal{Y}}(G, \mathcal{Y})$ is defined by
$$\delta_g^{\mathcal{Y}} \colon \mathcal{E}(G, \mathcal{Y}) \to \mathcal{Y}, \quad \langle \delta_g^{\mathcal{Y}}, \phi \rangle = \phi(g).$$
   If $\mathcal{Y} = \mathbb{C}$ then we denote simply $\delta_g^{\mathcal{Y}} = \delta_g$.

2. By using Remark 2.3, one can define a canonical linear mapping
$$\mathcal{E}'(G) \to \mathcal{E}'^{\mathcal{Y}}(G, \mathcal{Y}), \quad u \mapsto u \otimes \mathrm{id}_{\mathcal{Y}},$$
   which takes every distribution $u \colon \mathcal{E}(G) \to \mathbb{C}$ to its tensor product with the identity operator $\mathrm{id}_{\mathcal{Y}} \colon \mathcal{Y} \to \mathcal{Y}$.

Definition 2.3. *Let $u_1, u_2 \in \mathcal{E}'(G)$. Then the **tensor product of distributions** $u_1 \otimes u_2 \in \mathcal{E}'(G \times G)$ can be defined by using Remark 2.3 such that $\langle u_1 \otimes u_2, \phi_1 \otimes \phi_2 \rangle = \langle u_1, \phi_1 \rangle \cdot \langle u_2, \phi_2 \rangle$. On the other hand, there exists a continuous linear co-product*

$$\mathcal{E}(G) \to \mathcal{E}(G \times G), \quad \phi \mapsto \phi^{\Delta},$$

where $\phi^\Delta(x_1, x_2) := \phi(x_1 x_2)$ whenever $x_1, x_2 \in G$ and $\phi \in \mathcal{E}(G)$. The **convolution product of distributions** $u_1 * u_2 \in \mathcal{E}'(G)$ is defined by

$$(\forall \phi \in \mathcal{E}(G)) \quad \langle u_1 * u_2, \phi \rangle := \langle u_1 \otimes u_2, \phi^\Delta \rangle.$$

The convolution product makes the distribution space $\mathcal{E}'(G)$ into an associative algebra whose unit element is the Dirac distribution $\delta_\mathbf{1} \in \mathcal{E}'(G)$.

Example 2.2. Let us consider a few simple properties of the convolution product.

1. For arbitrary $g_1, g_2 \in G$ we have $\delta_{g_1} * \delta_{g_2} = \delta_{g_1 g_2}$.

2. For every $u_1, u_2 \in \mathcal{E}'(G)$ we have

$$\mathrm{supp}\,(u_1 * u_2) \subseteq \{x_1 x_2 \mid x_j \in \mathrm{supp}\, u_j \text{ for } j = 1, 2\}.$$

Definition 2.4. We shall think of $\mathfrak{g}$ as a real subalgebra of its **complexification** $\mathfrak{g}_\mathbb{C} := \mathbb{C} \otimes_\mathbb{R} \mathfrak{g}$, hence $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g}$. The **universal enveloping algebra** $\mathrm{U}(\mathfrak{g}_\mathbb{C})$ is the complex unital associative algebra satisfying the following conditions:

1. The complexification $\mathfrak{g}_\mathbb{C}$ is a Lie subalgebra of $\mathrm{U}(\mathfrak{g}_\mathbb{C})$.

2. For every complex unital associative algebra $\mathcal{A}$ and every linear mapping $\theta \colon \mathfrak{g}_\mathbb{C} \to \mathcal{A}$ satisfying $\theta([X, Y]) = \theta(X)\theta(Y) - \theta(Y)\theta(X)$ for all $X, Y \in \mathfrak{g}_\mathbb{C}$ there exists a unique extension of θ to a homomorphism of complex unital associative algebras $\mathrm{U}(\mathfrak{g}_\mathbb{C}) \to \mathcal{A}$.

One can prove that there always exists an algebra $\mathrm{U}(\mathfrak{g}_\mathbb{C})$ satisfying these conditions and it is uniquely determined up to an isomorphism of complex unital associative algebras. Moreover, there exists a unique (complex-)linear mapping $\mathrm{U}(\mathfrak{g}_\mathbb{C}) \to \mathrm{U}(\mathfrak{g}_\mathbb{C})$, $u \mapsto u^\perp$ such that

$$(vw)^\perp = w^\perp v^\perp, \quad (w^\perp)^\perp = w, \quad and \quad X^\perp = -X$$

for every $u, v \in \mathrm{U}(\mathfrak{g}_\mathbb{C})$ and $X \in \mathfrak{g}$ (see [15] for more details).

Example 2.3. If $\mathfrak{g}$ is an abelian Lie algebra of dimension n, then $\mathrm{U}(\mathfrak{g}_\mathbb{C})$ is the algebra of polynomials $\mathbb{C}[x_1, \ldots, x_n]$ and for every $p \in \mathbb{C}[x_1, \ldots, x_n]$ we have $p^\perp(x_1, \ldots, x_n) = p(-x_1, \ldots, -x_n)$.

We are going to describe in Remark 2.4 some realizations of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_\mathbb{C})$ which are needed later. To this end we first introduce the regular representations of G on distribution spaces.

Definition 2.5. *We shall use the* **translation maps** $\lambda_g \colon G \to G$, $x \mapsto gx$ *and* $\rho_g \colon G \to G$, $x \mapsto xg$ *defined by an arbitrary element* $g \in G$. *The corresponding* **regular representations** *of* G *on the distribution space* $\mathcal{D}'(G)$ *are defined by*

$$\lambda \colon G \to \operatorname{End}(\mathcal{D}'(G)), \quad \langle \lambda(g)u, \phi \rangle = \langle u, \phi \circ \lambda_g \rangle$$

and

$$\rho \colon G \to \operatorname{End}(\mathcal{D}'(G)), \quad \langle \rho(g)u, \phi \rangle = \langle u, \phi \circ \rho_{g^{-1}} \rangle$$

whenever $g \in G$, $u \in \mathcal{D}'(G)$, *and* $\phi \in \mathcal{D}(G)$. *For every* $X \in \mathfrak{g}$ *and* $\phi \in \mathcal{E}(G)$ *we also define the functions*

$$\dot{\lambda}(X)\phi \colon G \to \mathbb{C}, \quad (\dot{\lambda}(X)\phi)(z) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \phi(\exp_G(-tX)z)$$

and

$$\dot{\rho}(X)\phi \colon G \to \mathbb{C}, \quad (\dot{\rho}(X)\phi)(z) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \phi(z\exp_G(tX)).$$

Then we can define the **derivatives of the regular representations** *by*

$$\dot{\lambda} \colon \mathfrak{g} \to \operatorname{End}(\mathcal{D}'(G)), \quad \langle \dot{\lambda}(X)u, \phi \rangle := \langle u, \dot{\lambda}(-X)\phi \rangle$$

and

$$\dot{\rho} \colon \mathfrak{g} \to \operatorname{End}(\mathcal{D}'(G)), \quad \langle \dot{\rho}(X)u, \phi \rangle := \langle u, \dot{\rho}(-X)\phi \rangle.$$

These derivatives are homomorphisms of Lie algebras, hence condition (2) *in Definition 2.4 shows that they can be uniquely extended to unital homomorphisms of associative algebras* $\operatorname{U}(\mathfrak{g}_{\mathbb{C}}) \to \operatorname{End}(\mathcal{D}'(G))$. *These extensions will be also denoted by* $\dot{\lambda} \colon \operatorname{U}(\mathfrak{g}_{\mathbb{C}}) \to \operatorname{End}(\mathcal{D}'(G))$ *and* $\dot{\rho} \colon \operatorname{U}(\mathfrak{g}_{\mathbb{C}}) \to \operatorname{End}(\mathcal{D}'(G))$, *respectively.*

For later use, we also introduce the notation $\phi^{\perp}(x) := \phi(x^{-1})$ *for every* $\phi \in \mathcal{E}(G)$ *and* $x \in G$. *This gives rise to the idempotent linear mapping*

$$\mathcal{D}'(G) \to \mathcal{D}'(G), \quad u \mapsto u^{\perp},$$

where $\langle u^{\perp}, \phi \rangle := \langle u, \phi^{\perp} \rangle$ *for* $u \in \mathcal{D}'(G)$ *and* $\phi \in \mathcal{D}(G)$.

Remark 2.4. With Definition 2.5 at hand, we can describe some realizations of the universal enveloping algebra $\operatorname{U}(\mathfrak{g}_{\mathbb{C}})$ as follows. For the sake of simplicity, let us denote by $\mathcal{E}'_{\mathbf{1}}(G)$ the space of distributions on G with the support contained in $\{\mathbf{1}\}$, thought of as a complex unital associative algebra with respect to the convolution product, cf. Example 2.2. (This set should actually be denoted by $\mathcal{E}'_{\{\mathbf{1}\}}(G)$ according to Remark 2.2.) Recall that $\delta_{\mathbf{1}} \in \mathcal{E}'(G)$ is the Dirac distribution at $\mathbf{1} \in G$.

Both mappings

$$\operatorname{U}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{E}'_{\mathbf{1}}(G), \quad w \mapsto \dot{\lambda}(w)\delta_{\mathbf{1}},$$
$$\operatorname{U}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{E}'_{\mathbf{1}}(G), \quad w \mapsto \dot{\rho}(w)\delta_{\mathbf{1}}$$

are isomorphisms of complex unital associative algebras (see for instance Th. 1 in Sect. 10.4 of [36].) These isomorphisms are related by the commutative diagram

$$
\begin{array}{ccc}
\mathrm{U}(\mathfrak{g}_\mathbb{C}) & \xrightarrow{w \mapsto w^\perp} & \mathrm{U}(\mathfrak{g}_\mathbb{C}) \\
{\scriptstyle \dot{\lambda}(\cdot)\delta_\mathbf{1}} \downarrow & & \downarrow {\scriptstyle \dot{\rho}(\cdot)\delta_\mathbf{1}} \\
\mathcal{E}'_\mathbf{1}(G) & \xrightarrow{u \mapsto u^\perp} & \mathcal{E}'_\mathbf{1}(G)
\end{array}
$$

where the horizontal arrows stand for the mappings introduced in Definitions 2.4 and 2.5, respectively. From now on, *we perform the identification* $\mathrm{U}(\mathfrak{g}_\mathbb{C}) \simeq \mathcal{E}'_\mathbf{1}(G)$ *by means of the mapping* $w \mapsto \dot{\rho}(w)\delta_\mathbf{1}$, *by writing simply* w instead of $\dot{\rho}(w)\delta_\mathbf{1}$ whenever $w \in \mathrm{U}(\mathfrak{g}_\mathbb{C})$.

Proposition 2.1. *For every integer $m \geq 1$ and every compact neighbourhood K of $\mathbf{1} \in G$ there exist finitely many elements $u_1, \ldots, u_N \in \mathrm{U}(\mathfrak{g}_\mathbb{C})$ and the functions $\phi_1, \ldots, \phi_N \in \mathcal{C}_0^m(G)$ with $\bigcup\limits_{j=1}^{N} \operatorname{supp} \phi_j \subseteq K$ such that*

$$
\delta_\mathbf{1} = \sum_{j=1}^{N} \phi_j * u_j.
$$

Proof. Use Lemme 2 in [52] or Lemma 2.3 in [14]; see also the proof of Lemme 1.1 in [8]. □

2.2. Smooth vectors

Definition 2.6. *The **smooth vectors** for the representation $\pi\colon G \to \mathcal{B}(\mathcal{Y})$ are the elements of the linear subspace of $\mathcal{Y}$ defined by*

$$
\mathcal{Y}_\infty := \{y \in \mathcal{Y} \mid \pi(\cdot)y \in \mathcal{E}(G, \mathcal{Y})\}.
$$

The linear space $\mathcal{Y}_\infty$ will be endowed with the linear topology which makes the linear injective map

$$
\mathcal{Y}_\infty \to \mathcal{E}(G, \mathcal{Y}), \quad y \mapsto \pi(\cdot)y
$$

into a linear topological isomorphism onto its image.

For every distribution $u \in \mathcal{E}'(G)$ and every smooth vector $y \in \mathcal{Y}_\infty$ we then define

$$
\dot{\pi}(u)y := \langle u \widehat{\otimes} \mathrm{id}_\mathcal{Y}, \pi(\cdot)y \rangle \in \mathcal{Y}.
$$

Proposition 2.2. *The following assertions hold:*

1. *The space of smooth vectors $\mathcal{Y}_\infty$ is a Fréchet space and the inclusion map $\mathcal{Y}_\infty \hookrightarrow \mathcal{Y}$ is continuous.*

2. *The space $\mathcal{Y}_\infty$ is dense in $\mathcal{Y}$.*

3. For every $u \in \mathcal{E}'(G)$ we have $\dot{\pi}(u)\mathcal{Y}_\infty \subseteq \mathcal{Y}_\infty$.

4. The mapping $\dot{\pi} \colon \mathcal{E}'(G) \to \mathrm{End}\,(\mathcal{Y}_\infty)$ is a homomorphism of unital associative algebras.

5. For every $X \in \mathfrak{g}$ and $y \in \mathcal{Y}_\infty$ we have $\dot{\pi}(X)y := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi(\exp_G(tX))y$.

6. For every $y \in \mathcal{Y}_\infty$ we have a smooth mapping $\pi(\cdot)y \colon G \to \mathcal{Y}_\infty$.

Proof. See for instance [55] and Sect. 10.5 in [36]. $\qquad\square$

Notation 2.2. We shall always denote by $\mathcal{Y}_{-\infty}$ the space of continuous antilinear functionals on the Fréchet space $\mathcal{Y}_\infty$.

Proposition 2.3. *For every integer $m \geq 1$ there exist finitely many functions $\phi_1, \ldots, \phi_N \in \mathcal{C}_0^m(G)$ such that for every $y \in \mathcal{Y}_\infty$ there exist $y_1, \ldots, y_N \in \mathcal{Y}$ satisfying the equality $y = \dot{\pi}(\phi_1)y_1 + \cdots + \dot{\pi}(\phi_N)y_N$.*

Proof. Use Proposition 2.1 to get $u_1, \ldots, u_N \in \mathrm{U}(\mathfrak{g}_\mathbb{C})$ and $\phi_1, \ldots, \phi_N \in \mathcal{C}_0^m(G)$ with $\delta_{\mathbf{1}} = \sum_{j=1}^{N} \phi_j * u_j$. Then Proposition 2.2 shows that

$$y = \dot{\pi}(\delta_{\mathbf{1}})y = \sum_{j=1}^{N} \dot{\pi}(\phi_j)\dot{\pi}(u_j)y = \sum_{j=1}^{N} \dot{\pi}(\phi_j)y_j,$$

where we have denoted $y_j := \dot{\pi}(u_j)y$ for $j = 1, \ldots, N$. $\qquad\square$

Remark 2.5. As we already mentioned, the smooth vectors for representations of Lie groups were discussed in detail in [55]. Other important references in this connection are [19], [20], [36], [8], [12], [16], [51], and [11].

3. Smooth operators for unitary representations

We are going to discuss here the space of smooth operators for a given representation of a Lie group. The method of investigation was suggested in [49] and relies on exhibiting this space of operators as the space of smooth vectors for a suitable representation (see Definition 3.4). The main result is recorded as Theorem 3.1 and it is particularly significant in the case of unitary irreducible representations of nilpotent Lie groups (Corollary 3.1).

Notation 3.1. In this section we shall use the following notation:

- G is a connected unimodular Lie group with the Lie algebra $\mathfrak{g}$;

- $\mathrm{d}x$ denotes a fixed Haar measure on G;

- $\mathcal{H}$ is a complex Hilbert space;

- $\pi\colon G \to \mathcal{B}(\mathcal{H})$ is a continuous unitary representation, and $\mathcal{H}_\infty$ is the corresponding space of smooth vectors.

The following notion of smooth operator was singled out on page 349 in [29] and then further developed in [49].

Definition 3.1. *The set $\mathcal{B}(\mathcal{H})_\infty$ of **smooth operators** for the representation π is defined as the set of all operators $T \in \mathcal{B}(\mathcal{H})$ satisfying the following conditions:*

1. *We have $T(\mathcal{H}) + T^*(\mathcal{H}) \subseteq \mathcal{H}_\infty$.*

2. *For every $u \in \mathrm{U}(\mathfrak{g}_\mathbb{C})$ the operators $\dot\pi(u)T$ and $\dot\pi(u)T^*$ are bounded on $\mathcal{H}$.*

Example 3.1. For every $x, y \in \mathcal{H}_\infty$ the rank-one operator $(\cdot \mid x)y$ belongs to the space of smooth operators $\mathcal{B}(\mathcal{H})_\infty$. We shall see in Corollary 3.3 that the linear span of these rank-one operators is dense in $\mathcal{B}(\mathcal{H})_\infty$ provided that G is a nilpotent Lie group and π is an irreducible representation.

Remark 3.1. It follows at once by Definition 3.1 that $\mathcal{B}(\mathcal{H})_\infty$ is an associative $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

Definition 3.2. *We shall say that the representation π **has a smooth character** if for every $\phi \in \mathcal{D}(G)$ we have $\pi(\phi) \in \mathfrak{S}_1(\mathcal{H})$ and the linear mapping*

$$\mathcal{D}(G) \to \mathfrak{S}_1(\mathcal{H}), \quad \phi \mapsto \dot\pi(\phi)$$

*is continuous. In this case we define the corresponding **character** as*

$$\chi_\pi\colon \mathcal{D}(G) \to \mathbb{C}, \quad \chi_\pi(\phi) := \mathrm{Tr}\,\pi(\phi).$$

Note that $\chi_\pi \in \mathcal{D}'(G)$.

Example 3.2. Every unitary irreducible representation of a nilpotent Lie group has a smooth character; see for instance Th. 2 in §5 of Ch. II, Part.II in [50].

Remark 3.2. If the representation π has a smooth character, then there exists a continuous seminorm $p(\cdot)$ on $\mathcal{D}(G)$ such that

$$(\forall \phi \in \mathcal{D}(G)) \quad \|\dot\pi(\phi)\|_1 \le p(\phi).$$

In view of the definition of the topology on $\mathcal{D}(G)$ and of the fact that $\mathcal{D}(G)$ is dense in $\mathcal{C}_0^m(G)$ for every $m \ge 1$, it then easily follows that for every

compact subset $K \subseteq G$ there exists an integer $m \geq 1$ such that for every function $\phi \in \mathcal{C}_0^m(G) \cap \mathcal{E}'_K(G)$ we have $\dot{\pi}(\phi) \in \mathfrak{S}_1(\mathcal{H})$, and moreover the mapping

$$\mathcal{C}_0^m(G) \cap \mathcal{E}'_K(G) \to \mathfrak{S}_1(\mathcal{H}), \quad \phi \mapsto \dot{\pi}(\phi)$$

is linear and continuous.

Definition 3.3. *An **admissible ideal** is a non-trivial two-sided ideal $\mathcal{J}$ of $\mathcal{B}(\mathcal{H})$ with the following properties:*

1. *The ideal $\mathcal{J}$ is endowed with a complete norm $\|\cdot\|_{\mathcal{J}}$ such that for every $A, B \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{J}$ we have $\|ATB\|_{\mathcal{J}} \leq \|A\| \cdot \|T\|_{\mathcal{J}} \cdot \|B\|$ and $\|T^*\|_{\mathcal{J}} = \|T\|_{\mathcal{J}}$.*

2. *The ideal $\mathcal{F}(\mathcal{H})$ of finite-rank operators is a dense subspace of $\mathcal{J}$.*

3. *For every $x, y \in \mathcal{H}$ we have $\|(\cdot \mid x)y\|_{\mathcal{J}} = \|x\| \cdot \|y\|$.*

Example 3.3. Every Schatten ideal $\mathfrak{S}_p(\mathcal{H})$ with $1 \leq p \leq \infty$ is an admissible ideal. There exist many other examples of admissible ideals; see for instance [21].

Remark 3.3. Let $\mathcal{J}$ be an admissible ideal. By using condition (2) in Definition 3.3 with $A = \mathrm{id}_{\mathcal{H}}$ and $B = (\cdot \mid x)x$ for $x \in \mathcal{H}$, and then taking into account condition (3), it follows that $\|Tx\| \leq \|T\|_{\mathcal{J}} \cdot \|x\|$. That is, for every $T \in \mathcal{J}$ we have $\|T\| \leq \|T\|_{\mathcal{J}}$.

On the other hand, it follows at once by condition (3) in Definition 3.3 that for every $T \in \mathcal{F}(\mathcal{H})$ we have $\|T\|_{\mathcal{J}} \leq \|T\|_1$. Since $\mathcal{F}(\mathcal{H})$ is dense in $\mathfrak{S}_1(\mathcal{H})$, we get

$$(\forall T \in \mathfrak{S}_1(\mathcal{H})) \quad \|T\| \leq \|T\|_{\mathcal{J}} \leq \|T\|_1.$$

In particular, we have $\mathfrak{S}_1(\mathcal{H}) \subseteq \mathcal{J}$.

Definition 3.4. *For every admissible ideal $\mathcal{J}$ we define a linear representation $\pi_{\mathcal{J}}^{\otimes 2} \colon G \times G \to \mathcal{B}(\mathcal{J})$ by*

$$\pi_{\mathcal{J}}^{\otimes 2}(g_1, g_2)T := \pi(g_1)T\pi(g_2)^{-1}$$

for every $g_1, g_2 \in G$ and $T \in \mathcal{J}$.

Lemma 3.1. *The representation $\pi_{\mathcal{J}}^{\otimes 2} \colon G \times G \to \mathcal{B}(\mathcal{J})$ is continuous for every admissible ideal $\mathcal{J}$.*

Proof. If $T \in \mathcal{F}(\mathcal{H})$, then it is straightforward to check that the mapping $\pi_{\mathcal{J}}^{\otimes 2}(\cdot)T \colon G \times G \to \mathcal{J}$ is continuous.

Now let $T \in \mathcal{J}$ arbitrary. Since $\mathcal{J}$ is admissible, there exists a sequence $\{T_k\}_{k \geq 1}$ in $\mathcal{F}(\mathcal{H})$ such that $\lim_{k \to \infty} \|T - T_k\|_{\mathcal{J}} = 0$. On the other hand, since π is a unitary representation, it follows that for $k = 1, 2, \ldots$ and every pair $(g_1, g_2) \in G \times G$ we have

$$\|\pi_{\mathcal{J}}^{\otimes 2}(g_1, g_2)T - \pi_{\mathcal{J}}^{\otimes 2}(g_1, g_2)T_k\|_{\mathcal{J}} \leq \|T - T_k\|_{\mathcal{J}}.$$

Therefore $\pi_{\mathcal{J}}^{\otimes 2}(\cdot)T \colon G \times G \to \mathcal{J}$ is the uniform limit on $G \times G$ of the sequence of continuous mappings $\pi_{\mathcal{J}}^{\otimes 2}(\cdot)T$, hence it is in turn continuous. $\square$

Theorem 3.1. *Let $\pi \colon G \to \mathcal{B}(\mathcal{H})$ be a continuous unitary representation, assume that $\mathcal{J} \subseteq \mathcal{B}(\mathcal{H})$ is an admissible ideal, and denote by $\mathcal{J}_\infty$ the space of smooth vectors for the corresponding representation $\pi_{\mathcal{J}}^{\otimes 2}$. Then the following assertions hold:*

1. *We have $\mathcal{J}_\infty \subseteq \mathcal{B}(\mathcal{H})_\infty$.*

2. *If the Fréchet space of smooth vectors $\mathcal{H}_\infty$ is nuclear, then we have*

$$\mathcal{J}_\infty = \mathcal{B}(\mathcal{H})_\infty \subseteq \mathfrak{S}_1(\mathcal{H}),$$

   and the Fréchet space $\mathcal{J}_\infty$ does not depend on the choice of the admissible ideal $\mathcal{J}$.

Proof. (1) To prove the inclusion $\mathcal{J}_\infty \subseteq \mathcal{B}(\mathcal{H})_\infty$, let $T \in \mathcal{J}_\infty$ arbitrary, hence the mapping

$$G \times G \to \mathcal{J}, \quad (g_1, g_2) \mapsto \pi_{\mathcal{J}}^{\otimes 2}(g_1, g_2)T = \pi(g_1)T\pi(g_2)^{-1}$$

is smooth. In particular, the mapping $\pi(\cdot)T \colon G \to \mathcal{J}$ is smooth. On the other hand, it follows by Remark 3.3 that for arbitrary $x \in \mathcal{H}$ we have a continuous linear mapping $\mathcal{J} \to \mathcal{H}$, $T \mapsto Tx$. Hence the mapping $\pi(\cdot)Tx \colon G \to \mathcal{H}$ will be smooth as a composition of two smooth mappings. Thus for arbitrary $x \in \mathcal{H}$ we have $Tx \in \mathcal{H}_\infty$. Moreover, since the operation of taking the Hilbert space adjoint is ($\mathbb{R}$-linear and) continuous on $\mathcal{J}$ by condition (1) in Definition 3.3, it follows at once that $T^* \in \mathcal{J}_\infty$. Hence by the above reasoning with T replaced by T^* we get $T^*x \in \mathcal{H}_\infty$ for arbitrary $x \in \mathcal{H}$. Thus the operator T satisfies condition (1) in Definition 3.1. To check condition (2) in the same definition just note that since the mapping $\pi(\cdot)T \colon G \to \mathcal{J}$ is smooth, it follows that for every $u \in \mathrm{U}(\mathfrak{g}_{\mathbb{C}})$ we have $\dot\pi(u)T \in \mathcal{J}$, hence $\dot\pi(u)T \in \mathcal{B}(\mathcal{H})$. Since we have seen above that $T^* \in \mathcal{J}_\infty$, it also follows that $\dot\pi(u)T^* \in \mathcal{B}(\mathcal{H})$. This completes the proof of the fact that $T \in \mathcal{B}(\mathcal{H})_\infty$.
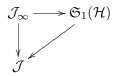
(2) If $\mathcal{H}_\infty$ is a nuclear space, then the inclusion map $\mathcal{H}_\infty \hookrightarrow \mathcal{H}$ is a nuclear operator (see Prop. 7.2 in Ch. III of [53]). Since condition (1) in Definition 3.1 shows that an arbitrary operator $T \in \mathcal{B}(\mathcal{H})_\infty$ factorizes as $\mathcal{H} \xrightarrow{T} \mathcal{H}_\infty \hookrightarrow \mathcal{H}$, it follows that $T \in \mathfrak{S}_1(\mathcal{H})$ (see Cor. 2 to Prop. 7.2 in Ch. III of [53]). Thus, by taking into account the above Assertion (1), we get

$$\mathcal{J}_\infty \subseteq \mathcal{B}(\mathcal{H})_\infty \subseteq \mathfrak{S}_1(\mathcal{H}). \tag{3.1}$$

To see that $\mathcal{J}_\infty$ does not depend on the choice of the admissible ideal $\mathcal{J}$, we shall prove the equality of Fréchet spaces

$$\mathcal{J}_\infty = \mathfrak{S}_1(\mathcal{H})_\infty, \tag{3.2}$$

where the right-hand side denotes the space of smooth vectors for the representation $\pi_{\mathfrak{S}_1(\mathcal{H})}^{\otimes 2} \colon G \times G \to \mathcal{B}(\mathfrak{S}_1(\mathcal{H}))$. First recall from Remark 3.3 that we have a continuous inclusion map $\mathfrak{S}_1(\mathcal{H}) \hookrightarrow \mathcal{J}$, which clearly intertwines the representations $\pi_{\mathfrak{S}_1(\mathcal{H})}^{\otimes 2}$ and $\pi_{\mathcal{J}}^{\otimes 2}$. It then easily follows by Definition 2.6 that we have a continuous inclusion map for the corresponding spaces of smooth vectors $\mathfrak{S}_1(\mathcal{H})_\infty \hookrightarrow \mathcal{J}_\infty$. On the other hand, we have already proved that $\mathcal{J}_\infty \subseteq \mathfrak{S}_1(\mathcal{H})$, hence there exists the following commutative diagram

$$\begin{array}{ccc} \mathcal{J}_\infty & \longrightarrow & \mathfrak{S}_1(\mathcal{H}) \\ \downarrow & \swarrow & \\ \mathcal{J} & & \end{array}$$

whose arrows stand for inclusion maps between Fréchet or Banach spaces. The arrows that point to $\mathcal{J}$ are continuous inclusions (by Remark 3.3 and Proposition 2.2(1)), hence the closed graph theorem implies that the inclusion map $\mathcal{J}_\infty \hookrightarrow \mathfrak{S}_1(\mathcal{H})$ is continuous as well. Since for arbitrary $T \in \mathcal{J}_\infty$ the mapping $G \times G \to \mathcal{J}_\infty$, $(g_1, g_2) \mapsto \pi(g_1)T\pi(g_2)^{-1}$ is smooth by Proposition 2.2(6), it then follows that the mapping $G \times G \to \mathfrak{S}_1(\mathcal{H})$, $(g_1, g_2) \mapsto \pi(g_1)T\pi(g_2)^{-1}$ is also smooth, hence $T \in \mathfrak{S}_1(\mathcal{H})_\infty$. Thus $\mathcal{J}_\infty = \mathfrak{S}_1(\mathcal{H})_\infty$ as sets. Since both sides of this equality are Fréchet spaces and we have already seen that the inclusion map $\mathfrak{S}_1(\mathcal{H})_\infty \hookrightarrow \mathcal{J}_\infty$ is continuous, it follows by the open mapping theorem that we have the equality of Fréchet spaces in (3.2).

Finally, note that for arbitrary $T \in \mathcal{B}(\mathcal{H})_\infty$ and every $u \in \mathrm{U}(\mathfrak{g}_\mathbb{C})$ we have $\dot{\pi}(u)T, \dot{\pi}(u)T^* \in \mathcal{B}(\mathcal{H})_\infty$ (see Definition 3.1). On the other hand, we have proved above that $\mathcal{B}(\mathcal{H})_\infty \subseteq \mathfrak{S}_1(\mathcal{H})$, hence $\dot{\pi}(u)T, \dot{\pi}(u)T^* \in \mathfrak{S}_1(\mathcal{H})$ for all $u \in \mathrm{U}(\mathfrak{g}_\mathbb{C})$, and this implies that $T \in \mathfrak{S}_1(\mathcal{H})_\infty$. Thus $\mathcal{B}(\mathcal{H})_\infty \subseteq \mathfrak{S}_1(\mathcal{H})_\infty$, and then by using Assertion (1) with $\mathcal{J} = \mathfrak{S}_1(\mathcal{H})$ we get $\mathcal{B}(\mathcal{H})_\infty = \mathfrak{S}_1(\mathcal{H})_\infty$. Now by (3.2) and (3.1) we get $\mathcal{J}_\infty = \mathcal{B}(\mathcal{H})_\infty \subseteq \mathfrak{S}_1(\mathcal{H})$, and this completes the proof. $\qquad\square$

Corollary 3.1. *Assume that G is a nilpotent Lie group and $\pi\colon G \to \mathcal{B}(\mathcal{H})$ is a unitary irreducible representation. If $\mathcal{J} \subseteq \mathcal{B}(\mathcal{H})$ is an admissible ideal, and we denote by $\mathcal{J}_\infty$ the space of smooth vectors for the corresponding representation $\pi_{\mathcal{J}}^{\otimes 2}$, then the following assertions hold:*

1. *The Fréchet space of smooth vectors $\mathcal{H}_\infty$ is nuclear.*

2. *We have $\mathcal{J}_\infty = \mathcal{B}(\mathcal{H})_\infty \subseteq \mathfrak{S}_1(\mathcal{H})$, and the Fréchet space $\mathcal{J}_\infty$ does not depend on the choice of the admissible ideal $\mathcal{J}$.*

3. *The space of smooth operators $\mathcal{B}(\mathcal{H})_\infty$ has the natural structure of a nuclear Fréchet space.*

Proof. Since the representation π is irreducible, there exists a linear topological isomorphism from the Fréchet space $\mathcal{H}_\infty$ onto the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^{d/2})$, where d is equal to the dimension of the coadjoint orbit of G corresponding to the representation π. (This follows by Th. 1 in §5 of Ch. II, Part. II in [50]; see also the Cor. to Th. 3.1 in [12], or [11].) On the other hand, it is well known that the Schwartz space $\mathcal{S}(\mathbb{R}^{d/2})$ is nuclear (see for instance Ex. 5 in §8 of Ch. III in [53]). Therefore the Fréchet space $\mathcal{H}_\infty$ is nuclear, and then Theorem 3.1 applies.

Finally, by using Assertion (2) when $\mathcal{J} = \mathfrak{S}_2(\mathcal{H})$ (the Hilbert-Schmidt ideal), it follows that $\mathcal{B}(\mathcal{H})_\infty$ is equal to the space of smooth vectors for the unitary representation $\pi_{\mathfrak{S}_2(\mathcal{H})}^{\otimes 2}$, hence it is a Fréchet space in a natural way. On the other hand, the representation $\pi_{\mathfrak{S}_2(\mathcal{H})}^{\otimes 2}$ is irreducible since so is π (see for instance the proof of Lemma 2.18(a) in [4]). Now the fact that $\mathcal{B}(\mathcal{H})_\infty$ is nuclear follows by the above Assertion (1) applied for the unitary irreducible representation $\pi_{\mathfrak{S}_2(\mathcal{H})}^{\otimes 2}\colon G \times G \to \mathcal{B}(\mathfrak{S}_2(\mathcal{H}))$. $\square$

Corollary 3.2. *If G is a nilpotent Lie group and $\pi\colon G \to \mathcal{B}(\mathcal{H})$ is a unitary irreducible representation, then the operators in $\mathcal{B}(\mathcal{H})_\infty$ are precisely the regularizing operators. That is, $A \in \mathcal{B}(\mathcal{H})_\infty$ if and only if A extends to a continuous linear map $A\colon \mathcal{H}_{-\infty} \to \mathcal{H}_\infty$, so that the diagram*

$$
\begin{array}{ccc}
\mathcal{H}_{-\infty} & \xrightarrow{\ A\ } & \mathcal{H}_\infty \\
\uparrow & & \downarrow \\
\mathcal{H} & \xrightarrow{\ A\ } & \mathcal{H}
\end{array}
$$

is commutative

Proof. By the closed graph theorem, it suffices to prove that if $A \in \mathcal{B}(\mathcal{H})_\infty$ and $f \in \mathcal{H}_{-\infty}$, then $Af \in \mathcal{H}_\infty$, in the sense that there exists a smooth vector denoted Af such that for every $\phi \in \mathcal{H}_\infty$ we have $(f \mid A^*\phi) = (Af \mid \phi)$. This is a consequence of the above Proposition 2.2(3), Corollary 3.1, and Th. 1.3(b) in [8].

Conversely, it follows by Definition 3.1 that the restriction to $\mathcal{H}$ of every continuous linear map $A\colon \mathcal{H}_{-\infty} \to \mathcal{H}_{\infty}$ belongs to $\mathcal{B}(\mathcal{H})_{\infty}$. $\qquad\square$

Corollary 3.3. *Assume that G is a nilpotent Lie group and $\pi\colon G \to \mathcal{B}(\mathcal{H})$ is a unitary irreducible representation. The linear space spanned by the operators $(\cdot \mid x)y$ with $x, y \in \mathcal{H}_{\infty}$ is dense in $\mathcal{B}(\mathcal{H})_{\infty}$.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})_{\infty}$ arbitrary. Then Corollary 3.1(2) shows that T is a smooth vector for the representation $\pi^{\otimes 2}_{\mathfrak{S}_2(\mathcal{H})}\colon G \times G \to \mathcal{B}(\mathfrak{S}_2(\mathcal{H}))$. It follows by Proposition 2.3 that there exist finitely many functions $\phi_1, \ldots, \phi_N \in \mathcal{C}_0^m(G \times G)$ and the operators $Y_1, \ldots, Y_N \in \mathfrak{S}_2(\mathcal{H})$ such that

$$T = \pi^{\otimes 2}_{\mathfrak{S}_2(\mathcal{H})}(\phi_1)Y_1 + \cdots + \pi^{\otimes 2}_{\mathfrak{S}_2(\mathcal{H})}(\phi_N)Y_N.$$

Since $\mathcal{D}(G \times G)$ is dense in $\mathcal{C}_0^m(G \times G)$ and $\mathcal{D}(G) \otimes \mathcal{D}(G)$ is dense in $\mathcal{D}(G \times G)$, it follows by Proposition 2.2(4) that T can be approximated in $\mathcal{B}(\mathcal{H})_{\infty}$ by finite linear combinations of operators of the form

$$\pi^{\otimes 2}_{\mathfrak{S}_2(\mathcal{H})}(\psi_1 \otimes \psi_2)Y = \pi(\psi_1)Y\pi(\psi_2^{\perp})$$

with $\psi_1, \psi_2 \in \mathcal{D}(G)$ and $Y \in \mathfrak{S}_2(\mathcal{H})$. On the other hand, such an Y can be approximated in $\mathfrak{S}_2(\mathcal{H})$ by finite linear combinations of operators $(\cdot \mid v_2)v_1$ with $v_1, v_2 \in \mathcal{H}$. The corollary now follows by noticing that

$$\pi(\psi_1)((\cdot \mid v_2)v_1)\pi(\psi_2^{\perp}) = (\cdot \mid \pi(\bar{\psi}_2^{\perp})v_2)\pi(\psi_1)v_1$$

and recalling that $\pi(\psi)v \in \mathcal{H}_{\infty}$ when $\psi \in \mathcal{D}(G)$ and $v \in \mathcal{H}$ ([19]). $\qquad\square$

Remark 3.4. Let $\mathcal{J} \subseteq \mathcal{B}(\mathcal{H})$ be any admissible ideal. If the representation π has a smooth character, then the corresponding space of smooth vectors $\mathcal{H}_{\infty}$ is nuclear according to Th. 2.6 in [8], hence the above Theorem 3.1(2) applies.

The inclusion $\mathcal{J}_{\infty} \subseteq \mathfrak{S}_1(\mathcal{H})$ can be alternatively proved in this case as follows. Let $T \in \mathcal{J}_{\infty}$ arbitrary. Since the representation π has a smooth character, we have a continuous linear mapping

$$\mathcal{D}(G) \to \mathfrak{S}_1(\mathcal{H}), \quad \phi \mapsto \dot{\pi}(\phi).$$

On the other hand, note that for every $\phi_1, \phi_2 \in \mathcal{D}(G)$ we have $\pi^{\otimes 2}_{\mathcal{J}}(\phi_1 \otimes \phi_2)T = \dot{\pi}(\phi_1)T\dot{\pi}(\phi_2^{\perp})$. Hence for arbitrary $Y \in \mathcal{J}$ we get a jointly continuous trilinear mapping

$$\mathcal{D}(G) \times \mathcal{D}(G) \times \mathcal{J} \to \mathfrak{S}_1(\mathcal{H}), \quad (\phi_1, \phi_2, Y) \mapsto \pi^{\otimes 2}_{\mathcal{J}}(\phi_1 \otimes \phi_2)Y,$$

which extends to a jointly continuous bilinear mapping

$$\mathcal{D}(G \times G) \times \mathcal{J} \to \mathfrak{S}_1(\mathcal{H}), \quad (\phi, Y) \mapsto \pi^{\otimes 2}_{\mathcal{J}}(\phi)Y$$

(see also Remark 2.3). By an argument similar to the one of Remark 3.2, we can further extend the above mapping to a continuous bilinear mapping

$$(\mathcal{C}_0^m(G \times G) \cap \mathcal{E}_K'(G \times G)) \times \mathcal{J} \to \mathfrak{S}_1(\mathcal{H}), \quad (\phi, Y) \mapsto \pi_{\mathcal{J}}^{\otimes 2}(\phi)Y \qquad (3.3)$$

for a suitable compact neighborhood K of $(\mathbf{1}, \mathbf{1}) \in G \times G$ and a suitably large integer $m \geq 1$. On the other hand, since $T \in \mathcal{J}_\infty$, it follows by Proposition 2.3 that there exist finitely many functions $\phi_1, \ldots, \phi_N \in \mathcal{C}_0^m(G \times G) \cap \mathcal{E}_K'(G \times G)$ and the operators $Y_1, \ldots, Y_N \in \mathcal{J}$ such that $T = \pi_{\mathcal{J}}^{\otimes 2}(\phi_1)Y_1 + \cdots + \pi_{\mathcal{J}}^{\otimes 2}(\phi_N)Y_N$, hence by (3.3) we get $T \in \mathfrak{S}_1(\mathcal{H})$.

4. Weyl-Pedersen calculus

In the present section we provide a brief discussion of the remarkable Weyl correspondence constructed in [49] and we shall also describe some complementary results which were recently obtained in [4].

4.1. Preduals for coadjoint orbits

This subsection records some properties of the coadjoint orbits of nilpotent Lie groups which play a crucial role for the construction of the Weyl-Pedersen calculus.

Setting 1. We shall use the following notation:

1. Let G be a connected, simply connected, nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then the exponential map $\exp_G \colon \mathfrak{g} \to G$ is a diffeomorphism with the inverse denoted by $\log_G \colon G \to \mathfrak{g}$.

2. We denote by $\mathfrak{g}^*$ the linear dual space to $\mathfrak{g}$ and by $\langle \cdot, \cdot \rangle \colon \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the natural duality pairing.

3. Let $\xi_0 \in \mathfrak{g}^*$ with the corresponding coadjoint orbit $\mathcal{O} := \mathrm{Ad}_G^*(G)\xi_0 \subseteq \mathfrak{g}^*$.

4. Let $\pi \colon G \to \mathcal{B}(\mathcal{H})$ be any unitary irreducible representations associated with the coadjoint orbit $\mathcal{O}$ by Kirillov's theorem (see [35]).

5. The *isotropy group* at ξ_0 is $G_{\xi_0} := \{g \in G \mid \mathrm{Ad}_G^*(g)\xi_0 = \xi_0\}$ with the corresponding *isotropy Lie algebra* $\mathfrak{g}_{\xi_0} = \{X \in \mathfrak{g} \mid \xi_0 \circ \mathrm{ad}_{\mathfrak{g}}X = 0\}$. If we denote the *center* of $\mathfrak{g}$ by $\mathfrak{z} := \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = \{0\}\}$, then it is clear that $\mathfrak{z} \subseteq \mathfrak{g}_{\xi_0}$.

6. Let $n := \dim \mathfrak{g}$ and fix a sequence of ideals in $\mathfrak{g}$,

$$\{0\} = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \cdots \subseteq \mathfrak{g}_n = \mathfrak{g}$$

such that $\dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1$ and $[\mathfrak{g}, \mathfrak{g}_j] \subseteq \mathfrak{g}_{j-1}$ for $j = 1, \ldots, n$.

7. Pick any $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$ for $j = 1, \ldots, n$, so that the set $\{X_1, \ldots, X_n\}$ will be a *Jordan-Hölder basis* in $\mathfrak{g}$.

Definition 4.1. *Consider the set of **jump indices** of the coadjoint orbit $\mathcal{O}$ with respect to the aforementioned Jordan-Hölder basis $\{X_1, \ldots, X_n\} \subseteq \mathfrak{g}$,*

$$e := \{j \in \{1, \ldots, n\} \mid \mathfrak{g}_j \not\subseteq \mathfrak{g}_{j-1} + \mathfrak{g}_{\xi_0}\} = \{j \in \{1, \ldots, n\} \mid X_j \notin \mathfrak{g}_{j-1} + \mathfrak{g}_{\xi_0}\}$$

*and then define the corresponding **predual of the coadjoint orbit** $\mathcal{O}$,*

$$\mathfrak{g}_e := \mathrm{span}\,\{X_j \mid j \in e\} \subseteq \mathfrak{g}.$$

We note the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_{\xi_0} \dot{+} \mathfrak{g}_e$.

Remark 4.1. Let $\{\xi_1, \ldots, \xi_n\} \subseteq \mathfrak{g}^*$ be the dual basis for $\{X_1, \ldots, X_n\} \subseteq \mathfrak{g}$. Then the coadjoint orbit $\mathcal{O}$ can be described in terms of the jump indices mentioned in Definition 4.1. More specifically, if we denote

$$\mathfrak{g}_{\mathcal{O}}^* := \mathrm{span}\,\{\xi_j \mid j \in e\} \quad \text{and} \quad \mathfrak{g}_{\mathcal{O}}^\perp := \mathrm{span}\,\{\xi_j \mid j \notin e\},$$

then the coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^* \simeq \mathfrak{g}_e^* \times \mathfrak{g}_e^\perp$ is the graph of a certain *polynomial mapping* $\mathfrak{g}_e^* \to \mathfrak{g}_e^\perp$. This leads to the following pieces of information on $\mathcal{O}$:

1. $\dim \mathcal{O} = \dim \mathfrak{g}_e = \mathrm{card}\, e =: d$;

2. if we let $j_1 < \cdots < j_d$ such that $e = \{j_1, \ldots, j_d\}$, then the mapping

$$\mathcal{O} \to \mathbb{R}^d, \quad \xi \mapsto (\langle \xi, X_{j_1} \rangle, \ldots, \langle \xi, X_{j_d} \rangle)$$

is a global chart which takes the Liouville measure of $\mathcal{O}$ to a Lebesgue measure on $\mathbb{R}^d$.

We define the Fourier transform $\mathcal{S}(\mathcal{O}) \to \mathcal{S}(\mathfrak{g}_e)$ by

$$(\forall X \in \mathfrak{g}_e) \quad \widehat{a}(X) = \int_{\mathcal{O}} \mathrm{e}^{-\mathrm{i}\langle \xi, X \rangle} a(\xi) \mathrm{d}\xi$$

for every $a \in \mathcal{S}(\mathcal{O})$, where $\mathrm{d}\xi$ stands for a Liouville measure on $\mathcal{O}$. This Fourier transform is invertible. The Lebesgue measure on $\mathfrak{g}_e$ can be normalized such that the Fourier transform extends to a unitary operator

$$L^2(\mathcal{O}) \to L^2(\mathfrak{g}_e), \quad a \mapsto \widehat{a},$$

and its inverse is defined by the usual formula. We shall always consider the predual $\mathfrak{g}_e$ endowed with this normalized measure (see for instance Lemma 1.6.1 in [48] and Lemma 4.1.1 in [49] for more details and proofs for the above assertions).

Remark 4.2. Some basic references for the geometry of coadjoint orbits of nilpotent Lie groups include [50], [46], [47], [48], and [11]; see also [5].

4.2. Weyl-Pedersen calculus and Moyal identities

We begin this subsection by the general construction of a Weyl correspondence due to [49].

Definition 4.2. *The **Weyl-Pedersen calculus** $\mathrm{Op}^\pi(\cdot)$ for the unitary representation π is defined for every $a \in \mathcal{S}(\mathcal{O})$ by*

$$\mathrm{Op}^\pi(a) = \int_{\mathfrak{g}_e} \widehat{a}(X)\pi(\exp_G X)\mathrm{d}X \in \mathcal{B}(\mathcal{H}).$$

*We call $\mathrm{Op}^\pi(a)$ the **pseudo-differential operator** with the symbol $a \in \mathcal{S}(\mathcal{O})$.*

Theorem 4.1. *The Weyl-Pedersen calculus has the following properties:*

1. *For every symbol $a \in \mathcal{S}(\mathcal{O})$ we have $\mathrm{Op}^\pi(a) \in \mathcal{B}(\mathcal{H})_\infty$ and the mapping $\mathcal{S}(\mathcal{O}) \to \mathcal{B}(\mathcal{H})_\infty$, $a \mapsto \mathrm{Op}^\pi(a)$ is a linear topological isomorphism.*

2. *For every $T \in \mathcal{B}(\mathcal{H})_\infty$ we have $T = \mathrm{Op}^\pi(a)$, where $a \in \mathcal{S}(\mathcal{O})$ satisfies the condition $\widehat{a}(X) = \mathrm{Tr}\left(\pi(\exp_G X)^{-1}T\right)$ for every $X \in \mathfrak{g}_e$.*

3. *For every $a, b \in \mathcal{S}(\mathcal{O})$ we have*

    (a) $\mathrm{Op}^\pi(\bar{a}) = \mathrm{Op}^\pi(a)^*$;

    (b) $\mathrm{Tr}\left(\mathrm{Op}^\pi(a)\right) = \int_{\mathcal{O}} a(\xi)\mathrm{d}\xi$;

    (c) $\mathrm{Tr}\left(\mathrm{Op}^\pi(a)\mathrm{Op}^\pi(b)\right) = \int_{\mathcal{O}} a(\xi)b(\xi)\mathrm{d}\xi$;

    (d) $\mathrm{Tr}\left(\mathrm{Op}^\pi(a)\mathrm{Op}^\pi(b)^*\right) = \int_{\mathcal{O}} a(\xi)\overline{b(\xi)}\mathrm{d}\xi$.

Proof. See Th. 4.1.4 and Th. 2.2.7 in [49]. $\square$

Definition 4.3. *Recall from Remark 3.1 that $\mathcal{B}(\mathcal{H})_\infty$ is an involutive associative subalgebra of $\mathcal{B}(\mathcal{H})$. It then follows by Theorem 4.1(1) that there exists an uniquely defined bilinear associative **Moyal product***

$$\mathcal{S}(\mathcal{O}) \times \mathcal{S}(\mathcal{O}) \to \mathcal{S}(\mathcal{O}), \quad (a, b) \mapsto a\#^\pi b$$

such that

$$(\forall a, b \in \mathcal{S}(\mathcal{O})) \quad \mathrm{Op}^\pi(a\#^\pi b) = \mathrm{Op}^\pi(a)\mathrm{Op}^\pi(b).$$

Thus $\mathcal{S}(\mathcal{O})$ is made into an involutive associative algebra such that the mapping $\mathcal{S}(\mathcal{O}) \to \mathcal{B}(\mathcal{H})_\infty$, $a \mapsto \mathrm{Op}^\pi(a)$ is an algebra isomorphism.

Notation 4.1. Recall that $\mathcal{H}_{-\infty}$ stands for the space of continuous antilinear functionals on $\mathcal{H}_{\infty}$ and the corresponding pairing will always be denoted by $(\cdot \mid \cdot)\colon \mathcal{H}_{-\infty} \times \mathcal{H}_{\infty} \to \mathbb{C}$. just as the scalar product in $\mathcal{H}$, since they agree on $\mathcal{H}_{\infty} \times \mathcal{H}_{\infty}$ if we think of the natural inclusions $\mathcal{H}_{\infty} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-\infty}$ (see for instance [8] for more details).

Definition 4.4. If $f \in \mathcal{H}_{-\infty}$ and $\phi \in \mathcal{H}_{\infty}$, or $f, \phi \in \mathcal{H}$, then we define the corresponding **ambiguity function**

$$\mathcal{A}(f, \phi) = \mathcal{A}_\phi f \colon \mathfrak{g}_e \to \mathbb{C}, \quad (\mathcal{A}_\phi f)(X) = (f \mid \pi(\exp_G X)\phi).$$

For $\phi \in \mathcal{H}_{-\infty}$ and $f \in \mathcal{H}_{\infty}$ we also define $(\mathcal{A}_\phi f)(X) = \overline{(\phi \mid \pi(\exp_G(-X))f)}$ whenever $X \in \mathfrak{g}_e$.

*It follows by Proposition 4.1(1) below that if $f, \phi \in \mathcal{H}$, then $\mathcal{A}_\phi f \in L^2(\mathfrak{g}_e)$, so we can use the aforementioned Fourier transform to define the corresponding **cross-Wigner distribution** $\mathcal{W}(f, \phi) \in L^2(\mathcal{O})$ such that $\widehat{\mathcal{W}(f, \phi)} := \mathcal{A}_\phi f$.*

The second equality in Proposition 4.1(1) below could be referred to as the *Moyal identity* since that classical identity (see for instance [22]) is recovered in the special case when G is a simply connected Heisenberg group.

Proposition 4.1. *The following assertions hold:*

1. *If $\phi \in \mathcal{H}$, then $\mathcal{A}_\phi f \in L^2(\mathfrak{g}_e)$. We have*

$$\begin{aligned}
(\mathcal{A}_{\phi_1} f_1 \mid \mathcal{A}_{\phi_2} f_2)_{L^2(\mathfrak{g}_e)} &= (f_1 \mid f_2)_{\mathcal{H}} \cdot (\phi_2 \mid \phi_1)_{\mathcal{H}} \\
&= (\mathcal{W}(f_1, \phi_1) \mid \mathcal{W}(f_2, \phi_2))_{L^2(\mathcal{O})}
\end{aligned} \tag{4.1}$$

   for arbitrary $\phi_1, \phi_2, f_1, f_2 \in \mathcal{H}$.

2. *If $\phi_0 \in \mathcal{H}$ with $\|\phi_0\| = 1$, then the linear operator $\mathcal{A}_{\phi_0}\colon \mathcal{H} \to L^2(\mathfrak{g}_e)$, $f \mapsto \mathcal{A}_{\phi_0} f$, is an isometry and we have*

$$\int\limits_{\mathfrak{g}_e} (\mathcal{A}_{\phi_0} f)(X) \cdot \pi(\exp_G X)\phi \, \mathrm{d}X = (\phi \mid \phi_0)f$$

   for every $\phi \in \mathcal{H}_{\infty}$ and $f \in \mathcal{H}$. In particular,

$$\int\limits_{\mathfrak{g}_e} (\mathcal{A}_{\phi_0} f)(X) \cdot \pi(\exp_G X)\phi_0 \, \mathrm{d}X = f$$

   for arbitrary $f \in \mathcal{H}$.

Proof. See [4]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Corollary 4.1. *The following assertions hold:*

1. *For each $a \in \mathcal{S}(\mathcal{O})$ we have*

$$(\mathrm{Op}^\pi(a)\phi \mid f)_{\mathcal{H}} = (\widehat{a} \mid \mathcal{A}_\phi f)_{L^2(\mathfrak{g}_e)} = (a \mid \mathcal{W}(f,\phi))_{L^2(\mathcal{O})}$$

   whenever $\phi, f \in \mathcal{H}$. Similar equalities hold if $a \in \mathcal{S}'(\mathcal{O})$ and $\phi, f \in \mathcal{H}_\infty$.

2. *If $\phi_1, \phi_2 \in \mathcal{H}_\infty$ and $a := \mathcal{W}(\phi_1, \phi_2) \in \mathcal{S}(\mathcal{O})$, then $\mathrm{Op}^\pi(a)$ is a rank-one operator, namely $\mathrm{Op}^\pi(a) = (\cdot \mid \phi_2)\phi_1$.*

Proof. See [4].   □

Assertion (3) in the following corollary in the special case of square-integrable representations reduces to a theorem of [10] and [13]. One thus recovers Th. 2.3 in [24] in the case of the Schrödinger representation of the Heisenberg group.

Corollary 4.2. *If $\phi_0 \in \mathcal{H}_\infty$ with $\|\phi_0\| = 1$, then the following assertions hold:*

1. *For every $f \in \mathcal{H}_{-\infty}$ we have*

$$\int\limits_{\mathfrak{g}_e} (\mathcal{A}_{\phi_0} f)(X) \cdot \pi(\exp_G X)\phi_0 \, \mathrm{d}X = f \qquad (4.2)$$

   where the integral is convergent in the weak*-topology of $\mathcal{H}_{-\infty}$.

2. *If $f \in \mathcal{H}_\infty$, then the above integral converges in the Fréchet topology of $\mathcal{H}_\infty$.*

3. *If $f \in \mathcal{H}_{-\infty}$, then we have $f \in \mathcal{H}_\infty$ if and only if $\mathcal{A}_{\phi_0} f \in \mathcal{S}(\mathfrak{g}_e)$.*

Proof. See [4].   □

Remark 4.3. Let $\mathcal{B}(\mathcal{H})^*_\infty$ be the dual of the Fréchet space $\mathcal{B}(\mathcal{H})_\infty$ and denote by $\langle \cdot, \cdot \rangle$ either of the duality pairings

$$\mathcal{B}(\mathcal{H})^*_\infty \times \mathcal{B}(\mathcal{H})_\infty \to \mathbb{C} \text{ and } \mathcal{S}'(\mathcal{O}) \times \mathcal{S}(\mathcal{O}) \to \mathbb{C}.$$

Then for every tempered distribution $a \in \mathcal{S}'(\mathcal{O})$ we can use Theorem 4.1(1) to define $\mathrm{Op}^\pi(a) \in \mathcal{B}(\mathcal{H})^*_\infty$ such that

$$(\forall b \in \mathcal{S}(\mathcal{O})) \quad \langle \mathrm{Op}^\pi(a), \mathrm{Op}^\pi(b) \rangle = \langle a, b \rangle.$$

Just as in Definition 4.2 we call $\mathrm{Op}^\pi(a)$ the *pseudo-differential operator* with the *symbol* $a \in \mathcal{S}'(\mathcal{O})$. Note that if actually $a \in \mathcal{S}(\mathcal{O})$, then the present notation agrees with Definition 4.2 because of Theorem 4.1(3c).

The continuity properties of the above pseudo-differential operators can be investigated by using modulation spaces of symbols; see [4] for details. Specifically, one can introduce modulation spaces $M_\phi^{r,s}(\pi)$ for every unitary irreducible representation $\pi\colon G \to \mathcal{B}(\mathcal{H})$. We always have $\mathcal{H}_\infty \subseteq M_\phi^{r,s}(\pi)$ and $M_\phi^{2,2}(\pi) = \mathcal{H}$. There exists a natural representation $\pi^\#\colon G \ltimes G \to \mathcal{B}(L^2(\mathcal{O}))$ such that for suitable $\Phi \in \mathcal{S}(\mathcal{O}) \setminus \{0\}$, the Weyl calculus $\mathrm{Op}^\pi(\cdot)$ defines a continuous linear mapping from the modulation space $M_\Phi^{\infty,1}(\pi^\#)$ into the space of *bounded* linear operators on $\mathcal{H}$. One of the main theorems of [23] is then recovered in the special case when π is the Schrödinger representation of the $(2n + 1)$-dimensional Heisenberg group. Some new results related to this circle of ideas will be established in Section 5 below.

Remark 4.4. (see [4]) The cross-Wigner distribution $\mathcal{W}(f_1, f_2) \in \mathcal{S}'(\mathcal{O})$ can be defined for arbitrary $f_1, f_2 \in \mathcal{H}_{-\infty}$ as follows. By using Corollary 3.2 we can define for $f_1, f_2 \in \mathcal{H}_{-\infty}$ the continuous antilinear functional

$$T_{f_1,f_2}\colon \mathcal{B}(\mathcal{H})_\infty \to \mathbb{C}, \quad T_{f_1,f_2}(A) := (f_1 \mid A f_2).$$

That is, $T_{f_1,f_2} \in \mathcal{B}(\mathcal{H})_\infty^*$, and then Th. 4.1.4(5) in [49] shows that there exists a unique distribution $a_{f_1,f_2} \in \mathcal{S}'(\mathcal{O})$ such that $\mathrm{Op}^\pi(a_{f_1,f_2}) = T_{f_1,f_2}$. Now define

$$\mathcal{W}(f_1, f_2) := a_{f_1,f_2}.$$

We can consider the rank-one operator $S_{f_1,f_2} := (\cdot \mid f_2)f_1\colon \mathcal{H}_\infty \to \mathcal{H}_{-\infty}$ and for arbitrary $A \in \mathcal{B}(\mathcal{H})_\infty$ thought of as a continuous linear map $A\colon \mathcal{H}_{-\infty} \to \mathcal{H}_\infty$ as above we have

$$\mathrm{Tr}\,(S_{f_1,f_2}A) = (f_1 \mid A f_2) = T_{f_1,f_2}(A).$$

Therefore, by using the trace duality pairing, we can identify the functional $T_{f_1,f_2} \in \mathcal{B}(\mathcal{H})_\infty^*$ with the rank-one operator $(\cdot \mid f_2)f_1$, and then we can write

$$(\forall f_1, f_2 \in \mathcal{H}_{-\infty}) \quad \mathrm{Op}^\pi(\mathcal{W}(f_1, f_2)) = (\cdot \mid f_2)f_1. \tag{4.3}$$

In particular, it follows that the above extension of the cross-Wigner distribution to a mapping $\mathcal{W}(\cdot, \cdot)\colon \mathcal{H}_{-\infty} \times \mathcal{H}_{-\infty} \to \mathcal{S}'(\mathcal{O})$ allows us to generalize the assertion of Corollary 4.1(2) to arbitrary $\phi_1, \phi_2 \in \mathcal{H}_{-\infty}$.

5. Modulation spaces

The modulation spaces play a central role in the time-frequency analysis (see [22]) and proved to be a very useful tool in the study of continuity properties of pseudo-differential operators ([23]). These classical ideas can be formulated within the representation theory of the Heisenberg groups, and this representation theoretic viewpoint turned out to be very effective in order

to extend the corresponding notions to unitary irreducible representations of arbitrary nilpotent Lie groups (see [4]). In the first two subsections of the present section we shall provide some preparations and then describe the general notion of modulation spaces introduced in [4]. We eventually illustrate this notion by discussing a specific class of irreducible representations on Hilbert spaces of the form $L^2(\mathcal{O})$, where $\mathcal{O}$ is any coadjoint orbit of a nilpotent Lie group (see Proposition 5.1 and Remark 5.2).

5.1. Semidirect products

Definition 5.1. *Let G_1 and G_2 be connected Lie groups and assume that we have a continuous group homomorphism $\alpha\colon G_1 \to \mathrm{Aut}\,G_2$, $g_1 \mapsto \alpha_{g_1}$. The corresponding* **semidirect product of Lie groups** *$G_1 \ltimes_\alpha G_2$ is the connected Lie group whose underlying manifold is the Cartesian product $G_1 \times G_2$ and whose group operation is given by*

$$(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, \theta_{h_1^{-1}}(g_2) h_2) \tag{5.1}$$

whenever $g_j, h_j \in G_j$ for $j = 1, 2$.

Let us denote by $\dot{\alpha}\colon \mathfrak{g}_1 \to \mathrm{Der}\,\mathfrak{g}_2$ the homomorphism of Lie algebras defined as the differential of the Lie group homomorphism $G_1 \to \mathrm{Aut}\,\mathfrak{g}_2$, $g_1 \mapsto T_1(\alpha_{g_1})$. Then the **semidirect product of Lie algebras** *$\mathfrak{g}_1 \ltimes_{\dot{\alpha}} \mathfrak{g}_2$ is the Lie algebra whose underlying linear space is the Cartesian product $\mathfrak{g}_1 \times \mathfrak{g}_2$ with the Lie bracket given by*

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], \dot{\alpha}(X_1)Y_2 - \dot{\alpha}(Y_1)X_2 + [X_2, Y_2]) \tag{5.2}$$

if $X_j, Y_j \in \mathfrak{g}_j$ for $j = 1, 2$. One can prove that $\mathfrak{g}_1 \ltimes_{\dot{\alpha}} \mathfrak{g}_2$ is the Lie algebra of the Lie group $G_1 \ltimes_\alpha G_2$ (see for instance Ch. 9 in [26]).

Remark 5.1. Let G_1 and G_2 be nilpotent Lie groups with a *unipotent automorphism* $\alpha\colon G_1 \to \mathrm{Aut}\,G_2$. That is, for every $X_1 \in \mathfrak{g}_1$ there exists an integer $m \geq 1$ such that $\dot{\alpha}(X_1)^m = 0$. Then an inspection of (5.2) shows that $\mathfrak{g}_1 \ltimes_{\dot{\alpha}} \mathfrak{g}_2$ is a nilpotent Lie algebra, hence $G_1 \ltimes_\alpha G_2$ is a nilpotent Lie group.

Example 5.1. Let G be a nilpotent Lie group. If we specialize Definition 5.1 for $G_1 := G$, $G_2 = (\mathfrak{g}, +)$, and $\alpha := \mathrm{Ad}_G\colon G \to \mathrm{Aut}\,\mathfrak{g}$, then we get the semidirect product $G \ltimes_{\mathrm{Ad}_G} \mathfrak{g}$ which is a nilpotent Lie group by Remark 5.1 and is isomorphic to the tangent group TG. The Lie algebra of $G \ltimes_{\mathrm{Ad}_G} \mathfrak{g}$ is $\mathfrak{g} \ltimes_{\mathrm{ad}_\mathfrak{g}} \mathfrak{g}_0$ (where $\mathfrak{g}_0$ stands for the abelian Lie algebra that has the same underlying linear space as $\mathfrak{g}$) and the corresponding exponential

map is given by

$$\exp_{G\ltimes_{\mathrm{Ad}_G}\mathfrak{g}}(X,Y) = (\exp_G X, \int_0^1 \mathrm{Ad}_G(\exp_G(sX))Y\,\mathrm{d}s)$$

$$= (\exp_G X, \int_0^1 \mathrm{e}^{s\cdot\mathrm{ad}_{\mathfrak{g}}X}Y\,\mathrm{d}s)$$

for every $(X,Y) \in \mathfrak{g} \ltimes_{\mathrm{ad}_{\mathfrak{g}}} \mathfrak{g}_0$ (see for instance Prop. 2.7(2) in [3]).

5.2. Modulation spaces for unitary representations

In this short subsection we just recall the definition of the modulation spaces for the unitary irreducible representations of nilpotent Lie groups. We refer to [4] for a more detailed discussion of this notion.

Definition 5.2. *Let* $\phi \in \mathcal{H}_\infty \setminus \{0\}$ *be fixed and assume that we have a direct sum decomposition* $\mathfrak{g}_e = \mathfrak{g}_e^1 \dotplus \mathfrak{g}_e^2$.
  Then let $1 \le r, s \le \infty$ *and for arbitrary* $f \in \mathcal{H}_{-\infty}$ *define*

$$\|f\|_{M_\phi^{r,s}} = \Big(\int_{\mathfrak{g}_e^2}\Big(\int_{\mathfrak{g}_e^1}|(\mathcal{A}_\phi f)(X_1,X_2)|^r\mathrm{d}X_1\Big)^{s/r}\mathrm{d}X_2\Big)^{1/s} \in [0,\infty]$$

with the usual conventions if r *or* s *is infinite. Then we call the space*

$$M_\phi^{r,s}(\pi) := \{f \in \mathcal{H}_{-\infty} \mid \|f\|_{M_\phi^{r,s}} < \infty\}$$

*a **modulation space** for the unitary representation* $\pi\colon G \to \mathcal{B}(\mathcal{H})$ *with respect to the decomposition* $\mathfrak{g}_e \simeq \mathfrak{g}_e^1 \times \mathfrak{g}_e^2$ *and the **window vector** $\phi \in \mathcal{H}_\infty \setminus \{0\}$.*

Example 5.2. For any choice of $\phi \in \mathcal{H}_\infty \setminus \{0\}$ in Definition 5.2 we have

$$M_\phi^{2,2}(\pi) = \mathcal{H}.$$

Indeed, this equality holds since $\|\mathcal{A}_\phi f\|_{L^2(\mathfrak{g}_e)} = \|\phi\| \cdot \|f\|$ for every $f \in \mathcal{H}$ (see Proposition 4.1 above).

5.3. A specific irreducible representation on $L^2(\mathcal{O})$

We are going to construct here some irreducible representations on the Hilbert spaces of the form $L^2(\mathcal{O})$, where $\mathcal{O}$ can be any coadjoint orbit of a nilpotent Lie group. A different construction involving the Moyal product (see Definition 4.3) was used in Def. 2.19 in the paper [4] in order to

get a representation $\pi^\#$ with the same representation space $L^2(\mathcal{O})$. The modulation spaces for $\pi^\#$ turned out to be relevant for establishing the continuity properties of the pseudo-differential operators obtained by the Weyl-Pedersen calculus for any unitary representation associated with the coadjoint orbit $\mathcal{O}$ (see also Remark 4.3).

Proposition 5.1. *Let Z be the center of the connected, simply connected, nilpotent Lie group G with the corresponding Lie algebra $\mathfrak{z} \subseteq \mathfrak{g}$. Endow the coadjoint orbit $\mathcal{O}$ with a Liouville measure and define*

$$\widetilde{\pi} \colon G \ltimes_{\mathrm{Ad}} \mathfrak{g} \to \mathcal{B}(L^2(\mathcal{O})), \quad (\widetilde{\pi}(g,Y)f)(\xi) = \mathrm{e}^{\mathrm{i}\langle \xi, Y \rangle} f(\mathrm{Ad}_G^*(g^{-1})\xi).$$

Then the following assertions hold:

1. *The group $\widetilde{G} := G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ is nilpotent and its center is $Z \times \mathfrak{z}$.*

2. *$\widetilde{\pi}$ is a unitary irreducible representation of $\widetilde{G}$.*

3. *Let us denote by $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes_{\mathrm{ad}_{\mathfrak{g}}} \mathfrak{g}_0$ the Lie algebra of $\widetilde{G}$ (where $\mathfrak{g}_0$ stands for the abelian Lie algebra with the same underlying linear space as $\mathfrak{g}$) and define*

$$\widetilde{X}_j = \begin{cases} (0, X_j) & for\ j = 1, \ldots, n, \\ (X_{j-n}, 0) & for\ j = n+1, \ldots, 2n. \end{cases}$$

Then $\widetilde{X}_1, \ldots, \widetilde{X}_{2n}$ is a Jordan-Hölder basis in $\widetilde{\mathfrak{g}}$ and the corresponding predual for the coadjoint orbit $\widetilde{\mathcal{O}} \subseteq \widetilde{\mathfrak{g}}^*$ associated with the representation $\widetilde{\pi}$ is

$$\widetilde{\mathfrak{g}}_{\widetilde{e}} = \mathfrak{g}_e \times \mathfrak{g}_e \subseteq \widetilde{\mathfrak{g}},$$

where $\widetilde{e}$ is the set of jump indices for $\widetilde{\mathcal{O}}$.

4. *The space of smooth vectors for the representation $\widetilde{\pi}$ is $\mathcal{S}(\mathcal{O})$.*

Proof. (1) Recall that the multiplication in the semi-direct product group $\widetilde{G}$ is given by

$$(g_1, Y_1) \cdot (g_2, Y_2) = (g_1 g_2, Y_1 + \mathrm{Ad}_G(g_1)Y_2)$$

while the bracket in the corresponding Lie algebra $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes_{\mathrm{ad}} \mathfrak{g}$ is defined by

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [X_1, Y_2] - [X_2, Y_1]). \tag{5.3}$$

An inspection of these equations quickly leads to the conclusion that $\widetilde{\mathfrak{g}}$ is a nilpotent Lie algebra with the center $\mathfrak{z} \times \mathfrak{z}$.

(2) If $(g_1, Y_1), (g_2, Y_2) \in \widetilde{G}$ and $f \in L^2(\mathcal{O})$, then for $\xi \in \mathcal{O}$ we have

$$
\begin{aligned}
\widetilde{\pi}(g_1, Y_1)(\widetilde{\pi}(g_2, Y_2)f)(\xi) &= \mathrm{e}^{\mathrm{i}\langle \xi, Y_1 \rangle}(\widetilde{\pi}(g_2, Y_2)f)(\mathrm{Ad}_G^*(g_1^{-1})\xi) \\
&= \mathrm{e}^{\mathrm{i}\langle \xi, Y_1 \rangle}\mathrm{e}^{\mathrm{i}\langle \mathrm{Ad}_G^*(g_1^{-1})\xi, Y_2 \rangle}f(\mathrm{Ad}_G^*(g_2^{-1})\mathrm{Ad}_G^*(g_1^{-1})\xi) \\
&= \mathrm{e}^{\mathrm{i}\langle \xi, Y_1 + \mathrm{Ad}_G(g_1)Y_2 \rangle}f(\mathrm{Ad}_G^*((g_1 g_2)^{-1})\xi)
\end{aligned}
$$

hence $\widetilde{\pi}(g_1, Y_1)\widetilde{\pi}(g_2, Y_2) = \widetilde{\pi}((g_1, Y_1)(g_2, Y_2))$. Next note that the representation $\widetilde{\pi}$ is unitary since the Liouville measure on $\mathcal{O}$ is invariant under the coadjoint action of G.

To see that $\widetilde{\pi}$ is irreducible, let $T \colon L^2(\mathcal{O}) \to L^2(\mathcal{O})$ be any bounded linear operator satisfying $T\widetilde{\pi}(g, Y) = \widetilde{\pi}(g, Y)T$ for arbitrary $(g, Y) \in \widetilde{G}$. We have to check that T is a scalar multiple of the identity operator on $L^2(\mathcal{O})$. By applying the assumption for $g = \mathbf{1} \in G$, we see that the operator T belongs to the commutant of the family of multiplication operators by the functions in the family $\{\mathrm{e}^{\mathrm{i}\langle Y, \cdot \rangle} \mid Y \in \mathfrak{g}\} \subseteq L^\infty(\mathcal{O})$. On the other hand, we recall that the mapping

$$
\mathcal{O} \to \mathbb{R}^d, \quad \xi \to (\langle \xi, X_{j_1} \rangle, \dots, \langle \xi, X_{j_d} \rangle)
$$

is a global chart which takes the Liouville measure of $\mathcal{O}$ to a Lebesgue measure on $\mathbb{R}^d$ (see for instance Remark 4.1). Then we can use the Fourier transform to see that the linear subspace generated by $\{\mathrm{e}^{\mathrm{i}\langle Y, \cdot \rangle} \mid Y \in \mathfrak{g}\}$ is weak*-dense in $L^\infty(\mathcal{O})$ ($\simeq L^1(\mathcal{O})^*$). Therefore the operator $T \colon L^2(\mathcal{O}) \to L^2(\mathcal{O})$ commutes with all the multiplication operators by functions in $L^\infty(\mathcal{O})$, and then it has to be in turn the multiplication operator by some function $\phi \in L^\infty(\mathcal{O})$. Now, by using the assumption that T commutes with $\pi(g, 0)$ for arbitrary $g \in G$, it easily follows that ϕ has to be a constant function since the coadjoint action of G on the orbit $\mathcal{O}$ is transitive.

(3) It is straightforward to check that $\widetilde{X}_1, \dots, \widetilde{X}_{2n}$ is a Jordan-Hölder basis in $\widetilde{\mathfrak{g}}$. Next note that $\mathcal{S}(\mathcal{O})$ is contained in the space of smooth vectors for the representation $\widetilde{\pi}$ and for arbitrary $f \in \mathcal{S}(\mathcal{O})$ and $(X, Y) \in \widetilde{\mathfrak{g}}$ we have

$$
(\forall \xi \in \mathcal{O}) \quad (\mathrm{d}\widetilde{\pi}(X, Y)f)(\xi) = \mathrm{i}\langle \xi, Y \rangle f(\xi) + \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\xi \circ \mathrm{e}^{t\mathrm{ad}_{\mathfrak{g}} X}). \tag{5.4}
$$

It then follows by a straightforward application of Prop. 2.4.1 in [46] and by Lemmas 1.4.1 and 1.5.1 in [48] that the set of jump indices for the coadjoint orbit $\widetilde{\mathcal{O}}$ is $\widetilde{e} = \{j_1, \dots, j_d, n + j_1, \dots, n + j_d\}$, and then $\widetilde{\mathfrak{g}}_{\widetilde{e}} = \mathfrak{g}_e \times \mathfrak{g}_e \subseteq \widetilde{\mathfrak{g}}$.

(4) It follows by (5.4) and by Lemmas 1.4.1 and 1.5.1 in [48] again that there exists a polynomial chart on $\mathcal{O}$ such that in the corresponding chart, the associative algebra generated by the image of $\mathrm{d}\widetilde{\pi}$ contains all the linear partial differential operators with polynomial coefficients. This implies that the space of smooth vectors for the representation $\widetilde{\pi}$ is equal to $\mathcal{S}(\mathcal{O})$, as asserted. $\qquad\square$

Corollary 5.1. *Assume the setting of Proposition 5.1. The ambiguity function*

$$\widetilde{\mathcal{A}} \colon L^2(\mathcal{O}) \times L^2(\mathcal{O}) \to L^2(\widetilde{\mathfrak{g}_{\widetilde{e}}}) = L^2(\mathfrak{g}_e \times \mathfrak{g}_e)$$

for the representation $\widetilde{\pi} \colon \widetilde{G} \to \mathcal{B}(L^2(\mathcal{H}))$ is given by the formula

$$(\widetilde{\mathcal{A}}_\phi f)(X,Y) = \int\limits_{\mathcal{O}} \mathrm{e}^{-\mathrm{i}\langle \xi, \int_0^1 \mathrm{e}^{s\cdot \mathrm{ad}_{\mathfrak{g}}X} Y\, \mathrm{d}s\rangle} f(\xi)\overline{\phi(\xi \circ \mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}X})}\mathrm{d}\xi$$

for arbitrary $X, Y \in \mathfrak{g}_e$ and $f, \phi \in L^2(\mathcal{O})$.

Proof. For every $f, h \in L^2(\mathcal{O})$ we have

$$(\widetilde{\mathcal{A}}_\phi f)(X,Y) = (f \mid \widetilde{\pi}(\exp_{\widetilde{G}}(X,Y))\phi)_{L^2(\mathcal{O})}. \tag{5.5}$$

On the other hand, for the element $(X,Y) \in \widetilde{\mathfrak{g}}$ we have

$$\exp_{\widetilde{G}}(X,Y) = (\exp_G X, \int\limits_0^1 \mathrm{Ad}_G(\exp_G(sX))Y\, \mathrm{d}s) = (\exp_G X, \int\limits_0^1 \mathrm{e}^{s\cdot \mathrm{ad}_{\mathfrak{g}}X}Y\, \mathrm{d}s)$$

(see Example 5.1) hence

$$(\widetilde{\pi}(\exp_{\widetilde{G}}(X,Y))\phi)(\xi) = \mathrm{e}^{\mathrm{i}\langle \xi, \int_0^1 \mathrm{e}^{s\cdot \mathrm{ad}_{\mathfrak{g}}X} Y\, \mathrm{d}s\rangle} \phi(\mathrm{Ad}_G^*(\exp_G(-X))\xi)$$

$$= \mathrm{e}^{\mathrm{i}\langle \xi, \int_0^1 \mathrm{e}^{s\cdot \mathrm{ad}_{\mathfrak{g}}X} Y\, \mathrm{d}s\rangle} \phi(\xi \circ \mathrm{e}^{\mathrm{ad}_{\mathfrak{g}}X})$$

and then the conclusion follows by (5.5). $\qquad\square$

Remark 5.2. Assume the setting of the above Proposition 5.1. It follows by Corollary 5.1 along with Schur's criterion for integral operators that there exists a constant $C_\Phi > 0$ such that for every $F \in L^2(\mathcal{O})$ and $Y \in \mathfrak{g}_e$ we have $\|\widetilde{\mathcal{A}}_\Phi F(\cdot, Y)\|_{L^2(\mathfrak{g}_e)} \le C_\Phi \|F\|_{L^2(\mathcal{O})}$, hence $\|F\|_{M_\Phi^{2,\infty}(\widetilde{\pi})} \le C_\Phi \|F\|_{L^2(\mathcal{O})}$. Therefore there exists a continuous inclusion map $L^2(\mathcal{O}) \hookrightarrow M_\Phi^{2,\infty}(\widetilde{\pi})$. See also [6] for similar inclusion maps for the modulation spaces in the setting of the magnetic Weyl calculus on nilpotent Lie groups.

Example 5.3. Assume that $\mathfrak{g}$ is two-step nilpotent Lie algebra, that is, we have $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = \{0\}$. Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be any nontrivial coadjoint orbit and pick $\xi_0 \in \mathcal{O}$. If we denote by $\mathfrak{z}$ the center of $\mathfrak{g}$, then

$$\mathcal{O} = \{\xi \in \mathfrak{g}^* \mid \quad \xi|_{\mathfrak{z}} = \xi_0|_{\mathfrak{z}}\},$$

since $\mathcal{O}$ is a flat orbit. Then by Corollary 5.1 along with the fact that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}$ we get

$$
\begin{aligned}
(\widetilde{\mathcal{A}}_\phi f)(X, Y) &= \int_{\mathcal{O}} \mathrm{e}^{-\mathrm{i}\langle \xi, \int_0^1 \mathrm{e}^{s \cdot \mathrm{ad}_\mathfrak{g} X} Y \, \mathrm{d}s\rangle} f(\xi) \overline{\phi(\xi \circ \mathrm{e}^{\mathrm{ad}_\mathfrak{g} X})} \mathrm{d}\xi \\
&= \int_{\mathcal{O}} \mathrm{e}^{-\mathrm{i}\langle \xi, Y + \frac{1}{2}[X,Y]\rangle} f(\xi) \overline{\phi(\xi + \xi \circ \mathrm{ad}_\mathfrak{g} X)} \mathrm{d}\xi \\
&= \mathrm{e}^{-\frac{\mathrm{i}}{2}\langle \xi_0, [X,Y]\rangle} \int_{\mathcal{O}} \mathrm{e}^{-\mathrm{i}\langle \xi, Y\rangle} f(\xi) \overline{\phi(\xi + \xi_0 \circ \mathrm{ad}_\mathfrak{g} X)} \mathrm{d}\xi.
\end{aligned}
$$

Using the above formula and suitable global coordinates on $\mathcal{O}$, one shows that the ambiguity function of the representation $\widetilde{\pi}$ agrees with the ambiguity function of the Schrödinger representation of a certain Heisenberg group, as defined in [22].

Acknowledgments

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