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Remarks on Bishop-type operators

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Dedicated to Professor Ion Colojoară on the occasion of his 80's anniversary

Abstract - We consider operators of weighted composition by irrational rotations on the space L^2 of the unit circle, for which several results of invariant subspaces are known. We show that various proofs of these results can be simplified, made more conceptual and slightly extended using certain topics on generalized spectral operators.

Key words and phrases : invariant subspace, irrational rotation.

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1. Introduction

Let $\alpha \in (0,1)$ be an irrational number and $T_\alpha : L^2[0,1) \to L^2[0,1)$ act on the space $L^2[0,1)$ of all classes of measurable, square integrable, complex-valued functions on the real interval $[0,1)$ by

$$(T_\alpha h)(x) = xh(\{x+\alpha\}), \qquad x \in [0,1)$$

where for any real number y the symbol $\{y\}$ denotes the fractional part of y, namely write $y = n + s$ with $n \in \mathbb{Z}$, $s \in [0,1)$ and set $\{y\} := s$. We can identify the interval $[0,1)$ and the unit circle $\mathbb{T} = \{e^{2\pi i x} \mid x \in [0,1)\}$ endowed with the normalized Lebesgue measure, so that $(T_\alpha h)(e^{2\pi i x}) = xh(e^{2\pi i(x+\alpha)})$ on the space $H := L^2(\mathbb{T})$. Then a corresponding Toeplitz operator is also induced on the Hardy space. The operators T_α were suggested in the 50's by E. Bishop as candidates to examples of operators acting on a complex Hilbert space H without closed linear invariant subspaces $\neq \{0\}, H$ – the existence of which is an open problem of operator theory. The main answer concerning these Bishop's operators was given in the 70's by Davie:

Theorem 1.1. (see [6]) *For almost every α, T_α does have closed, linear, nontrivial hyperinvariant subspaces.*

We call a subspace *hyperinvariant* if it is invariant under all bounded linear maps commuting with T_α on H – in particular, under T_α. Several extensions of this result exist, to more general weighted irrational rotations, on different function spaces, or in several variables, but the question remains open for a negligible set of highly transcendent numbers α.

The aim of this note is to show that various proofs of such results (mostly technical, based on an ingenious idea from [6]) can be simplified, made more conceptual and slightly extended by using certain topics from the work [3] by Colojoară & Foiaş on generalized spectral operators.

2. Preliminaries

Let us remind that, if $\alpha \in \mathbb{Q}$, then T_α easily proves to have (invariant) eigenspaces, while in the nontrivial case $\alpha \notin \mathbb{Q}$, T_α has no eigenvectors (see [6]). More general operators of the form

$$T_{\alpha,\varphi}h\,(x) = \varphi(x)h(e^{2\pi i(x+\alpha)}),$$

with $\varphi \in L^\infty(\mathbb{T})$ were then considered by MacDonald [13, 14], Blecher [2], Flattot [8], Chalendar, Partington & Pozzi [4, 5], on spaces $L^p(\mathbb{T})$ with $p \geq p_0(\alpha, \varphi)$ sufficiently large, on $\mathbb{T}$ as well as on the n-dimensional torus $\mathbb{T}^n$. Theorem 1.1 was successively extended to wider classes $\mathcal{A} \subset (0,1)$ of numbers α with $(0,1) \setminus \mathcal{A}$ negligible (but always, with $\mathcal{A} \neq (0,1)$) and of multipliers φ (including for instance all φ real analytic in the neighborhoods of $[0,1]$).

As we will prove, the operator theoretic part involved in the proofs can be expressed in terms of showing that both T_α and its Hilbert space adjoint T_α^* have quasiaffine representations by decomposable operators, see [3, Theorem II.4.5]. Also, uniform estimates in Birkhoff's ergodic theorem for the function $f := \ln|\varphi|$ and the action $f(e^{2\pi x}) \mapsto f(e^{2\pi(x+\alpha)})$ turn out to be necessary (which definitely limits the range of data (α, φ) that one can consider by these known techniques). This way we can recover some of the known proofs, and even obtain slightly new versions, like for instance the following statement.

Theorem 2.1. *Let $\alpha \in (0,1)$ be an arbitrary irrational number. Let $\varphi(x) = e^{f(x)}$, for $0 < x < 1$, where $f(x) = \sum_{k=-n}^{n} c_k e^{2k\pi i x}$ is an arbitrary trigonometric polynomial. Then $T_{\alpha,\varphi}$ has hyperinvariant subspaces.*

Under certain hypothesis (that hold a.e.) on the number theoretic properties of α and using some auxiliary ergodic results from [11, 12], we can prove as well versions of Theorem 2.1 for functions f with infinite trigonometric series, provided $c_k \to 0$ very rapidly as $|k| \to \infty$ (however, such conditions definitely are not fulfilled by $\varphi(x) = x$).

We start with the following known definition.

Definition 2.1. (see [10]) *The index* $\operatorname{ind}\alpha$ *of an irrational number* α *is the supremum of all* $l > 0$ *such that for any* $k > 0$ *there exist* p, q *with*

$$|\alpha - \frac{p}{q}| < \frac{k}{q^l}.$$

As it is well known by Liouville's theorem, if $\operatorname{ind}\alpha = \infty$ then α is transcendent. Recall also that by Dirichlet's theorem, for all irrational α we have $\operatorname{ind}\alpha \geq 2$. It's been proved by Roth that if α is algebraic irrational then $\operatorname{ind}\alpha = 2$. Also, Jarnik has shown that almost all numbers $\alpha \in (0,1)$ have finite index. For these topics we refer to [10].

Theorem 1.1 above holds for all irrational $\alpha \in (0,1)$ with $\operatorname{ind}\alpha < \infty$, and $\varphi(x) = x$ on L^2. The result was generalized in [13] to the case of those multipliers φ with $\ln|\varphi|$ well-approximable by step functions of intervals; also, for those φ with $\ln|\varphi| \in L^p$ piecewise monotone and $p > \operatorname{ind}\alpha$; in particular, for φ analytic in a neighbourhood of $[0,1]$ on spaces L^p with $1 < p < \infty$. The case $\varphi(x) = x^s$ was considered in [8] on L^2 for a larger class of α's including some non–Liouville numbers. Then a generalization of this was stated in [4] for products of two such Bishop type operators. Note that such a product is itself an operator of the same form, $T_{\alpha_1,\varphi_1}T_{\alpha_2,\varphi_2} = T_{\alpha_1+\alpha_2,\varphi}$. In [14], the existence of invariant subspaces was proved, for a certain class of α's, for Bishop operators in several variables. In the multivariable case we also mention [5], where the cyclic vectors of $T_{\alpha,\varphi}$ are described and the lack of eigenvectors is proved for certain φ. One can also try, for α arbitrary, to represent $T_{\alpha,\varphi} = e^{i\mathcal{D}}$ in terms of an infinitesimal generator $\mathcal{D} = \alpha \cdot \frac{1}{i}\frac{\partial}{\partial x} + M_\psi$ with $M_\psi h = \psi h$ for a function $\psi = \psi_{\alpha,\varphi}$ and use the semigroup structure. However, the existence of such representations was characterised in [15] by a cohomological obstruction that severely limits the range of φ (in particular, it does not hold for $\varphi(x) = x$).

3. Idea of the proof

We summarize in what follows, in a unified way, the main steps of the various proofs known so far to have provided invariant subspaces for Bishop type operators.

We start with the simplest case $\varphi(x) = x$.

Following [6], let us note $T := eT_\alpha$, whence the spectral radius $r(T)$ of T is 1, as follows from the formula

$$r(T_{\varphi,\alpha}) = e^{\int_0^1 \ln|\varphi(x)|\,dx},$$

which holds for a wide class of φ's, including the continuous ones on $[0,1)$ (see [13]). Generally, we set $T := r(T_\alpha)^{-1}T_\alpha$ and so $r(T) = 1$. Using the well known formula for the spectral radius $r(T) = \lim_{n\to\infty} \|T^n\|$, one can derive, briefly speaking, good estimates of the uniform operator norms $\|T^{\pm n}\|$ on suitable hilbertian spaces for large n, which leads to the existence of invariant subspaces by known techniques of Wermer (see [17]) and Atzmon (see [1]). A first obstacle to this aim is that generally T is not invertible. Moreover, using the technique mentioned above requires to deal with operators T having a rich functional calculus - almost unitaries, in some sense. For these (and other) reasons, a renorming of the space under consideration will be necessary, so that T extends to a more suitable (invertible etc) operator, say $\tilde{T} : \tilde{H} \to \tilde{H}$ on some hilbertian space $\tilde{H}$, that contains H densely. Strictly speaking, this extension $T \subset \tilde{T}$ will be a quasiaffine transformation of T (see [3, Definition II.4.1]), having the advantage that $\tilde{T}$ is decomposable in the sense of [3, Definition II.1.1].

The main tool in obtaining the existence of a rich functional calculus for $\tilde{T}$ is Denjoy-Carleman's theorem on quasi-analytic functions (see for example [13, 8, 4]). We remind below some known facts in this sense (see [3]).

Given a sequence of weights $\rho_n \geq 1$ where $n \in \mathbb{Z}$ such that $\rho_{n+m} \leq \rho_n \rho_m$ for all n, m and $\lim_{|n|\to\infty} \rho_n^{1/|n|} = 1$, the space of all continuous functions $f(e^{it}) = \sum_n c_n e^{int}$ on the unit circle such that

$$\|f\| := \sum_{n\in\mathbb{Z}} |c_n|\rho_n < \infty$$

becomes a Banach algebra $A_{(\rho_n)}$.

If Beurling's condition

$$\sum_{n\in\mathbb{Z}} \frac{\ln \rho_n}{n^2+1} < \infty$$

is verified (for example, if $\rho_n := |n|^{|n|^\rho}$, where $0 < \rho < 1$ is fixed), the algebra $A_{(\rho_n)}$ is regular. In particular, $A_{(\rho_n)}$ contains functions f, $g \not\equiv 0$ such that $fg \equiv 0$.

For an arbitrary complex Banach space X, let $B(X)$ denote the algebra of all bounded linear maps on X.

Definition 3.1. (see [3]) *Let $T \in B(X)$ be invertible. Set $\rho_n = \|T^n\|$ and $A_T := A_{(\rho_n)_n}$. We call T A_T-unitary if A_T is regular and the functional calculus $f \mapsto f(T)$ of T taking any polynomial $\sum_n c_n z^n$ into $\sum_n c_n T^n$ extends to a (unique) continuous morphism of algebras $A_T \ni f \mapsto f(T) \in B(X)$.*

Theorem 3.1. (see [17]) *If T is A_T-unitary, then it has invariant subspaces.*

Proof. One applies, roughly speaking, the multiplicativity property of the functional calculus of T, namely write that $f(T)g(T) = (fg)(T) = 0$, while $f(T) \neq 0$ and $g(T) \neq 0$. Then set $X_0 := \ker f(T)$. We have $TX_0 \subset X_0$, since $f(T)h = 0$ implies that $f(T)Th = Tf(T)h = 0$, too. We omit the details, that are known (see [17]).

For more general cases concerning α and φ, the present technique requires (see [13, 2, 8, 4]) a sharper, local version of Theorem 3.1 due to Atzmon [1].

4. Main results

We need certain topics on diophantine approximation. Remind that every irrational number $x \in (0,1)$ has a *continuous fraction* representation

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} \quad (a_1, a_2, a_3, \ldots \in \mathbb{N}).$$

That is, we write $\frac{1}{x} = a_1 + t_1$ with a_1 integer and $0 < t_1 < 1$, namely $a_1 = [\frac{1}{x}]$ and $t_1 = \{\frac{1}{x}\}$, then $\frac{1}{t_1} = a_2 + t_2$ with $a_2 \in \mathbb{N}$ and $t_2 \in (0,1)$, namely $a_2 = [\frac{1}{t_1}]$ etc, where $[y]$ stands as usual for the integer part of y, that is, $y = [y] + \{y\}$. By the formula $t_{n+1} = \{\frac{1}{t_n}\}$ for $n \geq 1$, it follows inductively that all $t_n = t_n(x)$ (and hence, all *partial quotients* $a_n = a_n(x)$) are measurable functions of $x \in (0,1)\setminus\mathbb{Q}$. Truncating the continued fraction of x at the n-th partial quotient a_n for each $n \geq 1$, we obtain the *convergents* $\frac{p_n}{q_n}$ of x

$$\frac{p_n}{q_n} := \frac{1}{a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_n}}} \quad (n \geq 1),$$

namely $\frac{p_1}{q_1} = \frac{1}{a_1}$, $\frac{p_2}{q_2} = \frac{1}{a_1 + \frac{1}{a_2}} = \frac{a_2}{a_1 a_2 + 1}$ etc, where $p_1 = 1$ and $q_1 = a_1$, $p_2 = a_2$ and $q_2 = a_1 a_2 + 1$ etc. Then $p_n = p_n(x) \geq 1$ and $q_n = q_n(x) \geq 1$ are also (integer-valued) measurable functions of x. For these topics we refer for instance to [7].

Theorem 4.1. (see [7]) *For every irrational number $x \in (0,1)$ we have $\lim_{n\to\infty} \frac{p_n}{q_n} = x$, and for every $n \geq 1$ the numbers p_n and q_n are relatively prime such that*

$$|x - \frac{p_n}{q_n}| < \frac{1}{q_n^2}$$

and

$$\frac{p_{2n}}{q_{2n}} < x < \frac{p_{2n-1}}{q_{2n-1}}.$$

Levy's Theorem 4.2 below is a nice application of the ergodic theory of numbers, providing us with a universal constant c such that $\sqrt[n]{q_n(x)} \to c$ almost everywhere with respect to x.

Theorem 4.2. (see [7]) *For almost all irrational $x \in (0,1)$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \ln q_n(x) = \frac{\pi^2}{12\ln 2}.$$

Now Theorem 4.2 leads to a slight strengthening of Dirichlet's theorem on rational approximation of irrational numbers, as follows below. In the sequel, let λ denote the Lebesgue measure.

Corollary 4.1. *For every ε_1, ε_2, $\mu \in (0,1)$ with $\varepsilon_1 < \varepsilon_2$, there exist a number $m_0 \geq 1$ and a measurable subset $M \subset (0,1)$ with $\lambda(M) > \mu$, such that for each natural number $m \geq m_0$ and every point $x \in M$ there are relatively prime integers $p \geq 1$ and $q \geq 1$ with*

$$0 < x - \frac{p}{q} < \frac{1}{q^2}$$

and

$$m^{1-\varepsilon_2} \leq q \leq m^{1-\varepsilon_1}.$$

Moreover, for m fixed we can select $p = p(x)$ and $q = q(x)$ $(x \in M)$ such that $p(\,\cdot\,)$ and $q(\,\cdot\,)$ are measurable functions.

Proof. Let ε_1, ε_2, $\mu \in (0,1)$ with $\varepsilon_1 < \varepsilon_2$. Set $c = \frac{\pi^2}{12\ln 2}$. Fix a positive $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2)$ sufficiently small such that

$$\frac{1-\varepsilon_1}{c+\varepsilon} - \frac{1-\varepsilon_2}{c-\varepsilon} > \frac{1}{2}\frac{\varepsilon_2 - \varepsilon_1}{c}. \tag{4.1}$$

By Levy's theorem above, the sequence of almost everywhere defined measurable functions $\frac{1}{n}\ln q_n$ is almost everywhere convergent to the constant function c. By Egorov's theorem, there exists a measurable set $M \subset (0,1)$ with $\lambda(M) > \mu$ such that $\frac{1}{n}\ln q_n \to c$ uniformly on M as $n \to \infty$. Let $n_0 \geq 1$ such that $\frac{1}{n}\ln q_n(x) \in (c-\varepsilon, c+\varepsilon)$ for all $n \geq n_0$ and almost all $x \in M$. Take $m_0 = \max(e^{\frac{(c+\varepsilon)(n_0+1)}{1-\varepsilon_1}}, e^{\frac{4c}{\varepsilon_2-\varepsilon_1}})$. Now let $m \geq m_0$ be arbitrary. Set $\nu = [\frac{\ln m^{1-\varepsilon_1}}{c+\varepsilon}]$. Since $m \geq m_0 \geq e^{\frac{(c+\varepsilon)(n_0+1)}{1-\varepsilon_1}}$, $\frac{\ln m^{1-\varepsilon_1}}{c+\varepsilon_1} - 1 \geq n_0$ and so $\nu - 1 \geq n_0$. If ν is even, let $n = \nu$; if ν is odd, let $n = \nu - 1$. In any case n is even and $n \geq n_0$. For every irrational $x \in M$, we may let $\frac{p}{q}$ be the n-th convergent of x, namely define $p := p_n(x)$ and $q = q_n(x)$.

By Dirichlet's theorem on rational approximation from above, $\frac{p}{q} < x$ and $x - \frac{p}{q} < \frac{1}{q^2}$. Using that $y - 1 \le [y]$ for $y = \frac{\ln m^{1-\varepsilon_1}}{c+\varepsilon}$, we obtain

$$\frac{\ln m^{1-\varepsilon_1}}{c+\varepsilon} - 2 \le \nu - 1 \le n \le \nu \le \frac{\ln m^{1-\varepsilon_1}}{c+\varepsilon}. \tag{4.2}$$

Since $m \ge e^{\frac{4c}{\varepsilon_2-\varepsilon_1}}$, we have $\ln m \ge \frac{4c}{\varepsilon_2-\varepsilon_1}$. By (4.1), this gives

$$\left(\frac{1-\varepsilon_1}{c+\varepsilon} - \frac{1-\varepsilon_2}{c-\varepsilon}\right)\ln m \ge 2$$

and so

$$\frac{\ln m^{1-\varepsilon_2}}{c-\varepsilon} \le \frac{\ln m^{1-\varepsilon_1}}{c+\varepsilon} - 2. \tag{4.3}$$

From (4.2) and (4.3) we derive

$$\frac{\ln m^{1-\varepsilon_2}}{c-\varepsilon} \le n \le \frac{\ln m^{1-\varepsilon_1}}{c+\varepsilon}.$$

Hence

$$m^{1-\varepsilon_2} \le e^{n(c-\varepsilon)} \;;\; e^{n(c+\varepsilon)} \le m^{1-\varepsilon_1}.$$

Since $n \ge n_0$, we have $c - \varepsilon \le \frac{1}{n}\ln q_n(x) \le c + \varepsilon$ for almost all $x \in M$, that is, $e^{n(c-\varepsilon)} \le q \le e^{n(c+\varepsilon)}$ almost everywhere. Then $m^{1-\varepsilon_2} \le q \le m^{1-\varepsilon_1}$.

If we aim to extend the results from the L^2 case to L^p spaces with $1 < p < \infty$, the following lemma (the idea of which is taken from [6]) is also needed.

Lemma 4.1. *Fix a real $p > 1$, a positive $\omega < 1 - \frac{1}{p}$ and a sequence of numbers $t_k > 0$ $(k \ge 1)$ with $\lim_{k\to\infty} t_k = 0$. Then for any $f \in L^p[0,1]$ nonnegative almost everywhere and any α, the sequence of sets*

$$E_k = \{x \in [0,1) \mid f(\{x - n\alpha\}) \le \frac{n^{1-\omega}}{t_k} \text{ for all } n \ge 1\} \quad (k \ge 1)$$

is increasing and satisfies $\lambda(\cup_k E_k) = 1$.

Proof. For every $k \ge 1$, one has the equality $[0,1) \setminus E_k = \cup_{n\ge1} M_{kn}$, where $M_{kn} = \{x \in [0,1) \mid f(\{x - n\alpha\}) > \frac{n^{1-\omega}}{t_k}\}$. Define the function $\sigma : \mathbb{R} \to [0,1)$ by $\sigma(y) = \{y\}$. A brief look at its graph shows that the restriction $\sigma|_I : I \to [0,1)$ is measure-preserving on any interval $I \subset \mathbb{R}$ of length one, that is, for every measurable subset m of the codomain $[0,1)$, the sets m and its preimage $\sigma^{-1}(m) := \{y \in I \,|\, \sigma(y) \in m\}$ have the same measure, $\lambda(\sigma^{-1}(m)) = \lambda(m)$. This property is inherited by every of the functions $\sigma_n : \mathbb{R} \to [0,1)$ defined by $\sigma_n(y) = \{y - n\alpha\}$, since they are translations of σ,

$\sigma_n(y) = \sigma(y - n\alpha)$. Then the map $\tau_n : [0,1) \to [0,1)$ defined by $\tau_n := \sigma_n|_{[0,1)}$ is measure-preserving. Consider the set $N_{kn} = \{s \in [0,1) \mid f(s) > \frac{n^{1-\omega}}{t_k}\}$. Note that M_{kn} is the preimage $\tau_n^{-1}(N_{kn}) := \{x \in [0,1) \mid \tau_n(x) \in N_{kn}\}$ of N_{kn} by τ_n, briefly: $M_{kn} = \tau_n^{-1}(N_{kn})$. Then

$$\lambda(M_{kn}) = \lambda(\tau_n^{-1}(N_{kn})) = \lambda(N_{kn}),$$

since τ_n is measure-preserving. It follows that for every k,

$$\lambda([0,1) \setminus E_k) = \lambda(\cup_n M_{kn}) \leq \sum_n \lambda(M_{kn}) = \sum_n \lambda(N_{kn}).$$

Now

$$(\frac{n^{1-\omega}}{t_k})^p \lambda(N_{kn}) \leq \int_{N_{kn}} f^p \, d\lambda \leq \|f\|_p^p.$$

Hence

$$\lambda([0,1) \setminus E_k) \leq \sum_n \|f\|_p^p (\frac{t_k}{n^{1-\omega}})^p = \|f\|_p^p \sum_n \frac{1}{n^{(1-\omega)p}} t_k^p = c \cdot t_k^p$$

for a constant $c = c_{f,p,\omega} < +\infty$, since $(1-\omega)p > 1$. Hence $\lambda([0,1) \setminus E_k) \to 0$ as $k \to \infty$.

A proof of Theorem 1.1. We give the promised version of the proof of Theorem 1.1 and add the necessary hints for the case of more general data φ and α (including for example those in Theorem 2.1). Generally, the results hold as usual for almost all α (generally, not all of them) and for suitable classes of weights φ. Part *(a)* of the proof compiles usual arguments from [6, 2, 13, 14, 8, 4], while part *(b)* applies results from [3, 9].

(a) Firstly, following [6], one extends $T_{\alpha,x}$ to the space L of all (classes of) Lebesgue measurable functions defined almost everywhere on $[0,1)$, so that T^{-n} also exists for $n = 1, 2, \ldots$, given by the formula

$$T^{-n} f(x) = F_n(x) f(\{x - \alpha n\}),$$

where

$$F_n(x) = \frac{e^{-n}}{\{x-\alpha\} \cdots \{x - n\alpha\}}$$

(we omit the details, that are routine). The computation in the case of an arbitrary $\varphi \in L^\infty(\mathbb{T})$ is straightforward, we just write $\varphi(\{x - j\alpha\})$ instead of $\{x - j\alpha\}$, respectively.

We represent now the interval $[0,1)$ as an increasing countable union $[0,1) = \cup_k E_{t_k}$ as in Lemma 4.1, that we apply for $f := |\ln|\varphi||$ and a suitable sequence $(t_k)_{k\geq 1}$ to obtain upper bounds of $|F_n|$ on E_{t_k} since the sets E_{t_k} have the form $E_{t_k} = \{x \mid \ \{x - n\alpha\}$ bounded from below $\}$.

Then we make use of Dirichlet's theorem in the form of Corollary 4.1, showing that for every natural integer n there exist two numbers p, q, relatively prime, such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$, $q \leq \sqrt{n}$. Moreover, for almost all α we can suppose that $q \geq n^{1/4}$. To this aim, note that we can exhaust almost all irrationals in $(0,1)$ by a countable net of measurable subsets $M = M_m$ as in the statement. Then simply suppose, by hypothesis, that $\alpha \in \cup_m M_m$. Then for our α in some fixed M, choose ε_1 and ε_2 suitably in order to get the desired estimates for the denominators q (of orders $\sqrt{n}$ and $\sqrt[4]{n}$, in the present case).

It follows that we have good approximations of the form $\{x - j\alpha\} \approx \{x - j\frac{p}{q}\}$ for $j = \overline{1,n}$, and hence, we can derive suitable estimates for the multipliers $F_n(x)$ on each of the sets E_t with $t = t_k$ for $k = 1, 2, \ldots$. Namely, we can get estimates for the essential supremum on E_{t_k}, of the form $\|F_n\|_{\infty, E_{t_k}} \leq n^{n^\rho}$, for some fixed positive $\rho < 1$ depending on ε_1 and ε_2. In the case $\varphi(x) = x$ this holds in an elementary way (see [6]): use the remarkable fact about the Dirichlet approximation $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$ that when j runs any partial set consisting of q consecutive integers, like $\{i+1, i+2, \ldots, i+q\}$, the factors $\{x - j\frac{p}{q}\}$ in the denominator of (the approximation of) F_q will run precisely the set $\{x - \frac{0}{q}\}$, $\{x - \frac{1}{q}\}$, $\{x - \frac{2}{q}\}$, $\ldots$, $\{x - \frac{q-1}{q}\}$, in some order that makes no matter, and so a factorial-type denominator appear, to be computed by Stirling's formula (or Γ functions). Then for an arbitrary, large n, we divide it by q and use $n = q \cdot m + r$ with $r < q$ to estimate, accordingly, $\|F_n\|$ in terms of $\|F_q\|, \|F_m\|$ and $\|F_r\|$. The estimate from below $q \geq \sqrt[4]{n}$ is to be used in conjunction with $m = \frac{n-r}{q} \leq \frac{n}{q}$ in order to find an estimate for m, and hence, for $\|F_m\|$.

For more general φ's such computations become complicated and a better way to unravel things is the ergodic theorem. Indeed, another idea behind most proofs seems to be the following (see [13]): once the multiplier F_n that appears in the formula $T^{-n} f(x) = F_n(x) f(\{x - \alpha n\})$ has a rather concrete form (in the case $\varphi(x) = x$ for instance), then estimating $|F_n(x)|$, from above and from below, is equivalent to finding estimates for the modulus of

$$\frac{1}{n} \ln |F_n(x)| \; = \; \int_0^1 F(y)\, dy \; - \; \frac{F(\tau(x)) + F(\tau^2(x)) + \cdots + F(\tau^n(x))}{n},$$

where $F(x) := \ln |\varphi(x)|$ $(= \ln x)$ and $\tau(z) := z e^{2\pi i \alpha}$ acts on the unit circle by a rotation of angle $2\pi\alpha$ (Weyl automorphism); to this aim, let us also remind the formula $r(T_{\varphi,\alpha}) = e^{\int_0^1 \ln |\varphi(y)| dy}$. An easy case for example is when $|\frac{1}{n}[F(x) + F(\tau(x)) + \cdots + F(\tau^{n-1}(x))] - \int_0^1 F(y)\, dy\,| \leq a_n$, where a_n is of order $\frac{1}{n^\varepsilon}$ as $n \to \infty$ for some $\varepsilon > 0$, which holds for very good weights φ (this is not the case for $\varphi(x) = x$, by the way). Such a convergence speed holds, for instance, for data like those in Theorem 2.1. Actually, it is enough to suppose there that the Fourier coefficients c_k of $f = \ln \varphi$ are sufficiently rapidly convergent to 0 as $|k| \to \infty$. Then we compute the trigonometric moments of the spectral measure $\langle E_U(\,\cdot\,)f, f\rangle$ of the unitary $Uh = h \circ \tau$ of composition with τ, localised on f, and apply the results in [11].

There are few concrete results on the speed of convergence $(a_n)_n$, for example let $F := \ln|\varphi|$ and write $F(e^{2\pi ix}) = \sum_k c_k e^{2\pi i\cdot kx}$; if $F \in L^1$ with $|c_k| \le \frac{ct}{k^{\mathrm{ind}\,\alpha+1+\varepsilon}}$, we may take $a_n = O(1/n)$ (see [12]); if $|c_k| \le \frac{ct}{k^{2+\varepsilon}}$, we may take $a_n = O(1/n^\varepsilon)$ using results in [11]; if α is well approximable by rationals, then τ is well approximable by periodic automorphisms, which also leads to an ergodic behaviour (see [16]).

However, as it is known, there cannot be a universal estimate of the speed of convergence in the ergodic theorem, even for a continuous function F. That is, by following only this technique there is no hope to prove existence of invariant subspaces for all $\alpha \in (0,1)$ and all $\varphi \in L^\infty(\mathbb{T})$. Moreover, there exist examples (see [14]) of Bishop-type operators with a bad behaviour of the sequence of norms $\|T^n\|$, so that the known techniques presented here can not lead to significant improvements with respect to [6, 13]. New ideas are then necessary in order to deal with the general case, more precisely with highly transcendent parameters α.

Returning to the manageable case when α is supposed to be within a suitable class $\mathcal{A} \subset (0,1)$ and φ behaves well, we obtain estimates of the form

$$\|T^{\pm n}\|_{L^2(E_t)} \le n^{n^\rho}.$$

The right hand side from above may be slightly larger, allowing to cover a wider class of α's (see [8, 4]) as long as Wermer / Atzmon -type conditions are fulfilled.

(b) In order to apply now the results we have mentioned from [3], we can take, for example, $\tilde{H} := \{f \mid \|f\|_{\tilde{H}} < \infty\}$, where for suitable positive constants c_k and t_k $(k \ge 1)$ with $t_k \to 0$ as $k \to +\infty$, the norm $\|\cdot\|_{\tilde{H}}$ is defined by

$$\|f\|_{\tilde{H}} := \sum_{k\ge 1} c_k \int_{E_{t_k}} \sum_{n\in\mathbb{Z}} |T^n f\,(x) e^{-|n|^\rho}|^2 dx$$

(integrals like those in the right hand side from above constantly appear in the proofs of such results, starting with the first one in [6]).

Using the estimates from above, of the form $\|T^{\pm n}\|_{L^2(E_t)} \le n^{n^\rho}$, we obtain

$$\|f\|_{\tilde{H}} \le ct.\, \|f\|_{L^2}.$$

Let $\tilde{T}$ denote the operator T acting on $\tilde{H}$. Then $\tilde{T}$ is $A_{\tilde{T}}$-unitary with spectrum $\sigma(\tilde{T}) =$ the unit circle. A suggestion to this aim is to start computing $\|\tilde{T}f\|_{\tilde{H}}$ for an arbitrary $f \in \tilde{H}$. Replace f by Tf in the formula defining $\|\cdot\|_{\tilde{H}}$, which after a shift $n+1 \mapsto n$ of the summation index $n \in \mathbb{Z}$ will alter the coefficient near $T^n f\,(x)$ by a factor $e^{|n+1|^\rho - |n|^\rho}$. Since $\rho < 1$, this factor is sufficiently negligible when checking Beurling's condition (in a simple case: if $\rho := 1/2$ and $n \geq 1$, this factor is $e^{\sqrt{n+1}-\sqrt{n}} = e^{(\sqrt{n+1}+\sqrt{n})^{-1}} \to 1$ as $n \to \infty$). Roughly speaking, this shows that $\tilde{T}$ is 'almost unitary'. Strictly speaking, it follows that $\tilde{T}$ is $A_{\tilde{T}}$-unitary, and so it is decomposable on $\tilde{H}$ (see [3]).

Let now $A : H\,(= L^2) \to \tilde{H}$ denote the inclusion $L^2 \subset \tilde{H}$. Then A is bounded with dense range. Also, $AT = \tilde{T}A$, that is, $T = A^{-1}\tilde{T}A$.

Definition 4.1. (see [3]) *Whenever an injective, bounded map $A : H \to \tilde{H}$ exists with dense range into another Hilbert space $\tilde{H}$ such that $T = A^{-1}\tilde{T}A$, we call T a* quasiaffine transformation *of $\tilde{T}$ and write $T < \tilde{T}$.*

The previous definition and the related results we are using hold as well in the context of the complex Banach spaces (see [3]). This makes it possible for us to deal as well with the case of the L^p spaces instead of $H = L^2$.

We have shown, so far, that

$$T < T_1$$

with T_1 decomposable and T_1 has the spectrum $= \mathbb{T}$ on some Banach space; namely, $T_1 = \tilde{T}$ on $\tilde{H}$.

The following properties of the quasiaffinity are straightforward (the symbol $*$ stands below, as usual, for the Banach space adjoint).

Proposition 4.1. (see [3]) *If $B < C$, then $C^* < B^*$. If B is A_B-unitary, then B^* is A_{B^*}-unitary.*

Now one easily checks that the Hilbert space adjoint $T^*_{\alpha,\varphi} = T_{-\alpha,\psi}$ of $T_{\alpha,\varphi}$ on H is also a Bishop-type operator, for a suitable $\psi = \psi_{\varphi,\alpha}$. Then we similarly obtain, by repeating from the scratch all reasoning above for $T_{-\alpha,\psi}$, that $T^* < \widetilde{T}$, where $\widetilde{T}$ is another $A_{\widetilde{T}}$-unitary operator with large spectrum ($\neq$ a single point) on some other Banach space. Hence by Proposition 4.1, we have $\widetilde{T}^* < T^{**}$ with $\widetilde{T}^* = A_{\widetilde{T}^*}$-unitary. Set $T_2 = \widetilde{T}^*$. Since H is a reflexive Banach space, $T^{**} = T$. Thus

$$T_2 < T < T_1$$

where T_i is A_{T_i}-unitary, for $i = 1, 2$.

The existence of a hyperinvariant subspace of T follows then from the theorem stated below.

Theorem 4.3. (see [3], see also [9]) *If $T_2 < T < T_1$ with $T_i = A_{T_i}$-unitary for $i = 1, 2$ and $\sigma(T_2) \neq$ a single point, then T has hyperinvariant subspaces.*

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