A Class of Partitionable Graphs with maximum number of edges

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Abstract. We call a graph G *O-graph* if there are an optimal coloring of the set of vertices of G and an optimal coloring of $\overline{G}$, the complement of G, such that any color-class of G intersects any color-class of $\overline{G}$. The main result of this paper is characterize this class by forbidden induced subgraphs.

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1 Introduction.

Throughout this paper $G = (V, E)$ is a simple(i.e. finite, undirected, without loops and multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. $\overline{G}$ designates the complement of G. A stable set in G is a set of mutually non-adjacent vertices, and the stability number of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set. The neighborhood of a vertex x is $N_G(x) = \{y \neq x | xy \in E\}$, denoted $N(x)$ when there is no ambiguity. The degree of x in G is $d_G(x) = |N_G(x)|$. If k is a positive integer, a k-coloring of G is any assignment $c : V-> \{1, ..., k\}$ with the property that for each $i \in \{1, ..., k\}$ the set $c^{-1}(i) = \{v|v \in V, c(v) = i\}$ is a stable set in G. The least possible number k of colors (the set $S_i = c^{-1}(i)$ is called the color class i of the coloring c) for which a graph G has a k-coloring is called the chromatic number of G and is denoted $\chi(G)$. By P_n, C_n and K_n we mean a chordless path on $n \geq 3$ vertices, the chordless cycle on $n \geq 3$ vertices, and the complete graph on $n \geq 1$ vertices. A clique in G is a subset A of $V(G)$ that induce a complete subgraph in G (that is a stable set in $\overline{G}$) The clique covering number of G (i.e. the chromatic number of $\overline{G}$) will be denoted by $\theta(G)$. The density or the clique number of G is the size of a largest clique in G, i.e., $\omega(G)$= $\alpha(\overline{G})$. A graph G is perfect if $\alpha(H)$=$\theta(H)$ (or, equivalently, $\chi(H)$=$\omega(H)$) holds for any induced subgraph H of G.

The quasi-cartesian product of the two graphs G_1=(V_1, E_1) and G_2=(V_2, E_2) is the graph $G_1 \otimes G_2$ whose vertex set is $V_1 \times V_2$ and two vertices (v_1, v_2) and

$(v_1\prime, v_2\prime)$ are adjacent iff:

$v_1 = v_1\prime$ and $v_2v_2\prime \in E_2$;

$v_1v_1\prime \in E_1$ and $v_2v_2\prime \in E_2$;

$v_1v_1\prime \in E_1$ and $v_2 = v_2\prime$.

Let us recall that in communication theory the determining of $\alpha(G^m)$, $(G^m := G \otimes G^{m-1}, m \geq 2)$, has an important role.

It is known (Olaru [8]) that for any two graphs G_1 and G_2 holds

$$\alpha(G_1 \otimes G_2) \leq min\{\alpha(G_1)\theta(G_2), \alpha(G_2)\theta(G_1)\},$$

and, consequently, if $\alpha(G) = \theta(G)$, then $\alpha(G \otimes H) = \alpha(G) \cdot \alpha(H)$, for any graph H, and so, for such graphs we have

$$\alpha(G^m) = [\alpha(G)]^m, m \geq 1.$$

A graph G is called $\alpha - partitionable$ if $\alpha(G) = \theta(G)$ holds.

The famous perfect graphs are, obviously, $\alpha - partitionable$ (V.Chvatal, R.L.Grahamm, A.F.Perold, S.H.Whiteside, [5]) and their characterization was gives by M.Chudnovsky, N.Robertson, P.D.Seymour, R.Thomas ([3]). Now is to find other class of $\alpha - partitionable$ graphs, that can be non perfect, e.g. $\alpha - partitionable$ graphs whose complement is $\alpha - partitionable$ too.

Let us call a graph G *partitionable* if $\theta(G) = \alpha(G)$ and $\chi(G) = \omega(G)$ holds.

We consider two questions which lead to perfect graphs.

We remind the following theorem of Lovasz ([6]), originally conjectured by Berge ([1]).

Perfect Graph Theorem. *A graph is perfect if and only if its complements is perfect.*

We will give another characterization of perfect graphs in terms of certain polytopes associated with graphs. Let G be a graph. The stable set polytope of G, also known as the vertex packing polytope of G, denoted by STAB(G), is the convex hull in $\mathbf{R}^{V(G)}$ of all incidence vectors of stable sets of G. A related polytope is the fractional stable set polytope or fractional vertex packing polytope QSTAB(G) $\subseteq \mathbf{R}^{\mathbf{V}(G)}$ defined by the constraints

$x_v \geq 0$ for every $v \in V(G)$,

$\sum_{v \in V(K)} x_v \leq 1$ for every clique K in G.

We have $STAB(G) \subseteq QSTAB(G)$. The following theorem implies the Perfect Graph Theorem.

Theorem ([7]). *For any graph G, the following conditions are equivalent.*

(i) G is perfect,

(ii) STAB(G)=QSTAB(G),

(iii) $\overline{G}$ *is perfect,*

(iv) STAB($\overline{G}$*)=QSTAB(*$\overline{G}$*).*

We remind a result of Chvatal ([4]). Let A be a 0,1 matrix. We say that the ith row of a matrix $A = (a_{ij})$ is undominated if there is no row index $j \neq i$ such that $a_{il} \leq a_{jl}$ for all l. Let G be a graph with $V(G) = \{v_1, v_2, ..., v_n\}$, and let $K_1, K_2, ..., K_m$ be its (inclusion-vise) maximal cliques. We define the maximal clique versus vertex incidence matrix of G to be the m x n matrix $A = (a_{ij})$, where $a_{ij} = 1$ if $v_j \in K_i$, and $a_{ij} = 0$ otherwise.

We consider the following linear program:

max $c \cdot x$ subject to $x \geq 0$ and $Ax \leq 1$.

For which matrices A is it true that for every objective function c, the linear program has integral optimum solution? It turns out that the answer to our question leads directly to perfect graphs.

Theorem ([4]). *The linear above program has an integral optimum solution for every objective function c if and only if the undominated rows of A form the maximal clique versus vertex incidence matrix of a perfect graph.*

2 Properties of O-graphs.

In the begining we give a characterization of O-graphs with the ω-partitionable graphs.

Definition 1. *A graph G is called an O-graph if there are an optimal coloring of G and an optimal coloring of $\overline{G}$, the complement of G, such that any color-class of G intersects any color-class of $\overline{G}$.*

We remind the following results.

Theorem 2. ([9]) *Let G be a graph with n vertices. Then G is an O-graph if and only if*

$$\chi(G) = \omega(G), \chi(\overline{G}) = \alpha(G) \text{ and } n = \alpha(G) \cdot \omega(G).$$

Remark 3. ([9]) *A graph G is an O-graph if and only if the set of vertices can be partitioned in ω stable sets each of it having α elements and in α cliques with ω vertices.*

Corollary 4. ([9]) *If G is an O-graph then, any color-class of any optimal coloring intersects any clique from any optimal covering with cliques of G.*

Definition 5. ([9]) *Let p, q be positive integer. A graph G is called $(p, q) -$ decomposable if G admits a p-coloring $(S_1, ..., S_p)$ where $|S_i| = q$ for all $i = 1, ..., p$.*

Corollary 6. ([9]) *A graph G is O-graph if and only if G is $(p, q) -$ decomposable and $\overline{G}$ is $(q, p) -$ decomposable , for some p and q.*

Proposition 7. ([9]) *Let G be an O-graph. Then for any optimal coloring $(S_1, ..., S_\omega)$ of G, the subgraph induced by $S_i \cup S_j$ has a perfect matching for all $i, j = 1, ..., \omega$ with $i \neq j$.*

Next, we characterize the class of Ograph by forbidden induced subgraphs.

A graph G is called $\omega - partitionable$ if $\overline{G}$ is $\alpha - partitionable$.

Definition. *A graf G with n vertices and m edges is called k-maximal (in relation to the number of edges), if its density equals k and any graph with n vertices and more that m edges has the density greater than k.*

Theorem 8. *Let $G = (V, E)$ be a graph with n vertices, m edges, $\alpha = \alpha(G)$ and $\omega = \omega(G)$. Then G is an O-graph with maximum number of edges if and only if G is $\omega - partitionable$ with $m = \alpha^2 \cdot C^2_\omega$.*

Proof. Let $G = (V, E)$ be a ω-partitionable graph with $m = \alpha^2 \cdot C_\omega^2$. Then there is an ω-coloring $(S_1, S_2, ..., S_\omega)$. So, $n \leq \alpha\omega$. We show that $d_G(x) = \alpha(\omega - 1)$, $\forall x \in V$, where $d_G(x)$ is the degree of x. If $\exists x_0 \in V$ such that $d_G(x_0) < \alpha(\omega - 1)$ then (because $d_G(x) \leq \alpha(\omega - 1)$, $\forall x \in V$) $m = \frac{1}{2}\sum_{x\in V} d_G(x) < \frac{1}{2}\alpha\omega\alpha(\omega - 1) = \alpha^2 C_\omega^2 = m$, a contradiction.

We show that $|S_i| = \alpha$, $\forall i = 1, ..., \omega$.

If $\exists i_0$ $(1 \leq i_0 \leq \omega)$ such that $|S_{i_0}| < \alpha$ then $\exists x_0 \in V - S_{i_0}$ such that $d_G(x_0) < \alpha(\omega - 1)$, a contradiction.

Because $\chi(G) = \omega(G)$ and $|S_i| = \alpha$, $\forall i = 1, ..., \omega$, it follows that $n = \alpha\omega$.

Because $|E(\overline{G})| = C_n^2 - m = \omega C_\alpha^2$, $\chi(G) = \omega$ and $n = \alpha\omega$, it follows that G admits a partition in ω α-stables with the property that two distinct vertices are adjacent if and only if they belong to distinct α-stables, that means that G is complete multipartite with ω parts each of them being α-stable, that means that G is ω-maximal. From Turan Theorem ([10], see [2]) it follows that G is O-graph with a maximum number of edges.

Reverse, let G be an O-graph with a maximum number of edges. Then G is ω-maximal and any vertex x from each ω-clique Q_j from the partition in α ω-cliques $C = (Q_1, Q_2, ..., Q_\alpha)$ is adjacent with exactly $\omega - 1$ vertices from any ω-clique of C. So the degree of x, $d_G(x) = \alpha(\omega - 1)$. So, $m = \frac{1}{2}\sum_{x\in V} d_G(x) = \frac{1}{2}n\alpha(\omega - 1)$ $= \alpha^2 C_\omega^2$, because $n = \alpha\omega$. Clearly, G is ω-partitionable.

Theorem 9. *Let G be a graph with n vertices, $\alpha = \alpha(G)$ and $\omega = \omega(G)$. Then G is an O-graph with maximum number of edges if and only if G is $\overline{K}_{1,2}$-free, $|E(G)| = \alpha^2 \cdot C_\omega^2$.*

Proof. Let G be an $\overline{K}_{1,2}$-free graph with $|E(G)| = \alpha^2 \cdot C_\omega^2$. Since G is a $\overline{K}_{1,2}$-free graph it follows that G is complete p-partite graph. Indeed. Let us partition the vertex set of G into classes of vertices with the same neighborhood. Since any two adjacent vertices have different neighborhoods (if $a, b \in V(G)$ and $ab \in E(G)$ then $b \in N_G(a)$ and $a \notin N_G(a)$), every class of the partition induces in G an empty graph. Now let x and y be two vertices from different classes, and suppose that x is not adjacent to y. Since x and y have different neighborhoods, there must exist a vertex z adjacent to one of them but non adjacent to another one. But then x, y, z induce in G a $\overline{K}_{1,2}$. This contradiction prove that the classes of the partition are the parts of a complete p-partite graph.

We show that G is O-graph with maximum number of edges. We know that $\overline{G}$ is disjoint reunion of p cliques and is q-partite. Let $(S_1, S_2, ...S_p)$ be a p-partition in stables of G with $|S_i| = s_i$, $1 \leq i \leq p$ and $(Q_1, Q_2, ...Q_q)$ a q-partition in stable of $\overline{G}$ with $|Q_j| = q_j$, $1 \leq j \leq q$. Since G is complete p-partite results that $p \leq \omega$ and $|E(\overline{G})| = \sum_{i=1}^{p} s_i \cdot (s_i - 1)/2$. Since S_i $(1 \leq i \leq p)$ are stables in G results also that $s_i \leq \alpha$ $(1 \leq i \leq p)$. Since $\omega \cdot C_\alpha^2 = |E(\overline{G})| = \sum_{i=1}^{p} s_i \cdot (s_i - 1)/2 \leqq \sum_{i=1}^{p} \alpha \cdot (\alpha - 1)/2 =$ $p \cdot \alpha \cdot (\alpha - 1)/2$ it results $\omega \leq p$. It follows that $p = \omega$. Since $\overline{G}$ is q-partite and is a disjoint reunion of p cliques results that $q \leq \alpha$ and $|E(G)| = \sum_{j=1}^{q} q_j \cdot (q_j - 1)/2 +$ $\sum_{i=1}^{q-1}\sum_{j=i+1}^{q} q_i \cdot (q_j - 1)$. Since Q_j $(1 \leq j \leq q)$ are stables in $\overline{G}$ results that $q_j \leq \omega$ $(1 \leq j \leq q)$. So $\alpha^2 \cdot C_\omega^2 = |E(G)| = \sum_{j=1}^{q} q_j \cdot (q_j - 1)/2 + \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} q_i \cdot (q_j - 1)$ $\leq \sum_{j=1}^{q} \omega \cdot (\omega - 1)/2 + \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} \omega \cdot (\omega - 1) = q \cdot C_\omega^2 + q \cdot (q - 1) \cdot C_\omega^2 = q^2 \cdot C_\omega^2$.

So $\alpha^2 \leq q^2$. It follows that $q = \alpha$. So G admits an ω-partition $(S_1, ..., S_\omega)$ in stables and $\overline{G}$ admits an α-partition $(Q_1, ..., Q_\alpha)$ in stables (in $\overline{G}$). Since $p = \omega$ and $\omega \cdot C_\alpha^2 = |E(\overline{G})| = \sum_{i=1}^{p} s_i \cdot (s_i - 1)/2$ it follows that $\omega \cdot C_\alpha^2 = \sum_{i=1}^{\omega} s_i \cdot (s_i - 1)/2$. So $s_i = |S_i| = \alpha$ $(1 \leq i \leq \omega)$ otherwise it would exists a stable S_i with $|S_i| > \alpha$. Since $q = \alpha$ and $\alpha^2 \cdot C_\omega^2 = |E(G)| = \sum_{j=1}^{q} q_j \cdot (q_j - 1)/2 + \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} q_i \cdot (q_j - 1)$ it follows that $\alpha^2 \cdot C_\omega^2 = \sum_{j=1}^{\alpha} q_j \cdot (q_j - 1)/2 + \sum_{i=1}^{\alpha-1} \sum_{j=i+1}^{\alpha} q_i \cdot (q_j - 1)$. So $q_j = |Q_j| = \omega$ $(1 \leq j \leq \alpha)$, otherwise it would exists a clique Q_j with $|Q_j| > \omega$. So G is (ω, α)-decomposable and $\overline{G}$ is (α, ω)-decomposable. So G is O-graph. G has a maximum number of edges, because otherwise it would exists a clique of cardinal $> \omega$ (because it would exists a vertex a ω-clique Q_j adjacent all vertices a ω-cliques Q_k, $k \neq j$).

Suppose that G is a O-graph with maximum number of edges. Since G is $(\omega, \alpha) - decomposable$ graph with maximum number of edges it follows that G is a $\overline{K}_{1,2} - free$. Indeed. Any three vertices induce in G either $\overline{K}_3$ (if all of them are in the same part) or $K_{1,2}$ (if two of them are in one part) or K_3 (if all the vertices are in different parts). Thus G does not contains $\overline{K}_{1,2}$ as an induced subgraph. We show that $|E(G)| = \alpha^2 \cdot C_\omega^2$. We know that $\overline{G}$ is a disjoint reunion of ω cliques of cardinality α. So $|E(\overline{G})| = \omega \cdot C_\alpha^2$. Since $n = \alpha \cdot \omega$ and $|E(G)| = C_n^2 - |E(\overline{G})|$ it follows that $|E(G)| = \alpha^2 \cdot C_\omega^2$.

Theorem 10. *Let G be a connected graph with n vertices, $\alpha = \alpha(G)$ and $\omega = \omega(G)$. Then G is an O-graph with maximum number of edges if and only if G is (P_4, Paw)-free, $|E(G)| = \alpha^2 \cdot C_\omega^2$.*

Proof. Let G be an O-graph with maximum number of edges. Then G is complete ω-partite with the parts of the same cardinality α and $\overline{G}$ is a disjoint reunion of ω cliques of α cardinal. So G is connected, (P_4, Paw)-free, $|E(\overline{G})| = \omega \cdot C_\alpha^2$ and $|E(G)| = C_n^2 - |E(\overline{G})| = \alpha^2 \cdot C_\omega^2$.

Let G be connected, (P_4, Paw)-free, $|E(G)| = \alpha^2 \cdot C_\omega^2$. We show that G is $\overline{K}_{1,2}$-free. Suppose that vertices a, b, c induce in G a $\overline{K}_{1,2}$ $(ac \in E(G))$. Let P_{ab} be a shortest path linking a to b in G. Since P_4 is forbiden, this path is exactly of length two. Let d be the unique internal vertex of the path. If d is adjacent to c, then a, b, c, d induce a Paw. If d is not adjacent to c, then a, b, c, d induce a P_4. Therefore G does not contains $\overline{K}_{1,2}$ as an induced subgraph. From Theorem 9 it follows that G is an O-graph with maximum of edges.

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