Sufficient Optimality Conditions for Nonlinear Programming with Mixed Constraints and Generalized ρ-Locally Arcwise Connected Functions

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Abstract. We derive sufficient optimality conditions for a nonlinear programming problem with inequality and equality constraints, where the functions involved are ρ-locally arcwise connected, ρ-locally Q-connected, ρ-locally P-connected and locally PQ-connected (notion introduced in this paper) and differentiable with respect to an arc. Sufficient optimality conditions are obtained in terms of the right differentials with respect to an arc of the functions.

Keywords : Generalized convexity;nonlinear programming; sufficient optimality conditions

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1 Introduction

Mathematical programs involving generalized convexity have been the subject of extensive study in the recent literature. Optimality conditions and duality results were obtained for such problems.

Ortega and Rheinboldt [9] introduced connected functions by taking the values of the functions on continuous curves (called "arcs") joining two points x and y, instead of the line segment joining these points. Avriel and Zang [1] called them "arcwise connected" functions.

Kaul *et al.* [6] defined locally arcwise connected sets which include arcwise connected sets [1] and locally starshaped sets [2]. Also, they introduced locally connected functions and locally Q-connected functions on a locally connected set and studied some local-global minimum properties satisfied by such functions. Kaul and Lyall [5] defined locally P-connected functions and studied properties of these functions and of locally connected (Q-connected) functions based on the concept of right differentiability of a function with respect to an arc. Results regarding the solution of nonlinear programming problem involving locally P-connected functions and sufficient optimality criteria for such a programming problem are also derived. These results are extended to the multiple objective programming by Lyall *et al.* [7] which have obtained Fritz John type necessary optimality criteria for non-linear programs and formulated a Mond-Weir type dual together with weak and strong duality results. A proper weak minimum is defined and duality results are established by using this concept.

Vial [16] defined ρ-convex functions. The generalization of ρ-convex functions to ρ-pseudo convex and ρ-quasi convex functions was given by Jeyakumar [3, 4].

In [10], Preda and Niculescu defined ρ-locally arcwise connected, ρ-locally Q-connected and ρ-locally P-connected functions and gave necessary and sufficient optimality conditions for a minimax optimization problem involving such functions. They considered a generalized Mond-Weir dual problem and established a duality result. In another paper [11], they considered a nonlinear multiple objective programming problem, gave necessary and sufficient optimality conditions and Wolfe and Mond-Weir type duality results. In [13], Stancu-Minasian considered a nonlinear fractional programming problem where the functions are ρ-locally arcwise connected, ρ-locally Q-connected and ρ-locally P-connected and obtained necessary and sufficient optimality conditions. A dual was formulated and duality results are proved.

Niculescu [8], considered a multiobjective programming problem in which the objective function contains a support function. A dual problem was formulated and a weak duality theorem was established under generalized ρ-local conectedness conditions. In [15] (see, also and [14]) Stancu-Minasian and Andreea Mădălina Stancu obtained sufficient optimality conditions for a nonlinear programming problem with inequality constraints and generalized ρ-locally arcwise connected functions.

In this paper we generalize the results obtained in [14] and [15] at the case of mixed constraints. We derive sufficient optimality conditions for a nonlinear programming problem with inequality and equality constraints where the functions involved are ρ-locally arcwise connected, ρ-locally Q-connected, ρ-locally P-connected, locally PQ-connected and differentiable with respect to an arc. Sufficient optimality conditions are obtained in terms of the right differentials with respect to an arc of the functions.

The organization of the remainder of this paper is as follows. In Section 2, we shall introduce the notation and definitions which are used throughout the paper. In Section 3, we shall give sufficient optimality criteria for a nonlinear programming problem with mixed constraints.

2 Preliminaries

In this section we introduce the notation and definitions which are used throughout the paper.

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space and $\mathbf{R}^n_+$ its nonnegative orthant, i.e., $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n, x_j \geqq 0,\ j = 1, \ldots, n\}$. Throughout the paper, the following conventions for vectors in $\mathbf{R}^n$ will be followed :

$x > y$ if and only if $x_i > y_i$ $(i = 1, \ldots, n)$,

$x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, n)$,

$x \geq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, n)$, but $x \neq y$.

Throughout the paper, all definitions, theorems and corollaries are numbered consecutively in a single numeration system in each section.

Let $X^0 \subseteq \mathbf{R^n}$ be a nonempty and compact subset of $\mathbf{R^n}$.

Let $\overline{x}, x \in X^0$. A continuous mapping $H_{\overline{x},x} : [0,1] \to \mathbf{R^n}$ with

$$H_{\overline{x},x}(0) = \overline{x}, H_{\overline{x},x}(1) = x$$

is called an arc from $\overline{x}$ to x.

Definition 2.1 [*6*] *We say that the set $X^0 \subseteq \mathbf{R^n}$ is a locally arcwise connected set at $\overline{x}$ ($\overline{x} \in X^0$) (X^0 is LAC($\overline{x}$), for short) if for any $x \in X^0$ there exist a positive number $a(x,\overline{x})$, with $0 < a(x,\overline{x}) \leqq 1$,and a continuous arc $H_{\overline{x},x}$ such that $H_{\overline{x},x}(\lambda) \in X^0$ for any $\lambda \in (0, a(x,\overline{x}))$.*

We say that the set X^0 is locally arcwise connected if X^0 is locally arcwise connected at any $x \in X^0$.

If we choose the function $H_{\overline{x},x}$ of the form $H_{\overline{x},x}(\lambda) = (1-\lambda)\overline{x} + \lambda x$, we find out the definition of locally starshaped set as given by Ewing [2].

Definition 2.2 [*10*]. *Let $f : X^0 \longrightarrow \mathbf{R}$ be a function, where $X^0 \subseteq \mathbf{R^n}$ is a locally arwise connected set at $\overline{x} \in X^0$ with the corresponding function $H_{\overline{x},x}(\lambda)$ and a maximum positive number $a(x,\overline{x})$ satisfying the required conditions. Also let $\rho \in \mathbf{R}$ and $d(\cdot,\cdot) : X^0 \times X^0 \to \mathbf{R}_+$ such that $d(x,\overline{x}) \neq 0$ for $x \neq \overline{x}$. We say that f is:*

(i_1) *ρ-locally arcwise connected at $\overline{x}$ (f is ρ-LCN($\overline{x}$), for short) if for any $x \in X^0$, there exist a positive number $d(x,\overline{x}) \leqq a(x,\overline{x})$ and an arc $H_{\overline{x},x}$ in X^0 on $[0, d(x,\overline{x})]$such that*

$$f(H_{\overline{x},x}(\lambda)) \leqq \lambda f(x) + (1-\lambda) f(\overline{x}) - \rho\lambda d(x,\overline{x}),\ 0 \leq \lambda \leq d(x,\overline{x}).(2.1)$$

(i_2) *ρ-locally Q-connected at $\overline{x}$(ρ-LQCN($\overline{x}$)) if for any $x \in X^0$, there exist a positive number $d(x,\overline{x}) \leqq a(x,\overline{x})$ and an arc $H_{\overline{x},x}$ in X^0 on $[0, d(x,\overline{x})]$ such that*

$$\left.\begin{array}{l} f(x) \leqq f(\overline{x}), \\ 0 \leq \lambda \leq d(x,\overline{x}) \end{array}\right\} \Rightarrow f(H_{\overline{x},x}(\lambda)) - f(\overline{x}) \leqq -\rho\lambda d(x,\overline{x}).$$

(i$_3$) *ρ-locally P-connected at $\overline{x}$(ρ-LPCN($\overline{x}$)) if for any $x \in X^0$, there exist a positive number $d(x,\overline{x}) \leqq a(x,\overline{x})$ and an arc $H_{\overline{x},x}$ in X^0 on $[0,d(x,\overline{x})]$ and a positive number $\gamma_{\overline{x},x}$ such that*

$$\left.\begin{array}{l} f(x) < f(\overline{x}), \\ 0 \leq \lambda \leq d(x,\overline{x}) \end{array}\right\} \Rightarrow f(H_{\overline{x},x}(\lambda)) \leqq f(\overline{x}) - \lambda\gamma_{\overline{x},x} - \rho\lambda d(x,\overline{x}).$$

(i$_4$) *ρ-locally strictly P-connected at $\overline{x}$(ρ-LSTPCN($\overline{x}$)) if for any $x \in X^0$, there exist a positive number $d(x,\overline{x}) \leqq a(x,\overline{x})$,an arc $H_{\overline{x},x}$ in X^0 on $[0,d(x,\overline{x})]$ and a positive number $\gamma_{\overline{x},x}$ such that*

$$\left.\begin{array}{l} x \neq \overline{x}, f(x) < f(\overline{x}), \\ 0 \leq \lambda \leq d(x,\overline{x}) \end{array}\right\} \Rightarrow f(H_{\overline{x},x}(\lambda)) < f(\overline{x}) - \lambda\gamma_{\overline{x},x} - \rho\lambda d(x,\overline{x}).$$

The function f is said to be ρ-locally strictly arcwise connected at $\overline{x} \in X^0$ (ρ-LSCN($\overline{x}$)) if for each $x \in X^0, x \neq x^0$ the inequality (2.1) is strict.

If f is ρ-LCN($\overline{x}$) (ρ-LSCN($\overline{x}$)) at each $\overline{x} \in X^0$, then f is said to be ρ-LCN (ρ-LSCN) on X^0.

If f is ρ-LQCN at each $\overline{x} \in X^0$, then f is said to be ρ-LQCN on X^0.

If f is ρ-LPCN at each $\overline{x} \in X^0$, then f is said to be ρ-LPCN on X^0.

Definition 2.3 [5]. *Let $f : X^0 \longrightarrow \mathbf{R}$ be a function, where $X^0 \subseteq \mathbf{R^n}$ is a locally arcwise connected set at $\overline{x} \in X^0$, with the corresponding function $H_{\overline{x},x}(\lambda)$ and a maximum positive number $a(x,\overline{x})$ satisfying the required conditions. The right differential of f at $\overline{x}$ with respect to the arc $H_{\overline{x},x}(\lambda)$ is give by*

$$(\mathrm{d}f)^+(\overline{x}, H_{\overline{x},x}(0^+)) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}[f(H_{\overline{x},x}(\lambda)) - f(\overline{x})]$$

provided the limit exists.

In this case, f is called right directionally differentiable at $\overline{x}$ with respect to the arc $H_{\overline{x},x}(\lambda)$.

If f is differentiable at any $\overline{x} \in X^0$, then f is said to be differentiable on X^0.

According to Avriel and Zang [1], $(\mathrm{d}f)^+(\overline{x}, H_{\overline{x},x}(0^+))$ may also be called directional derivative of f with respect to the arc $H_{\overline{x},x}(\lambda)$ at $\lambda = 0$. If the function f possess a right derivative with respect to the arc $H_{\overline{x},x}(\lambda)$ at $\lambda = 0$, then

$$f\left(H_{\overline{x},x}(\lambda)\right) = f(\overline{x}) + \lambda(\mathrm{d}f)^+(\overline{x}, H_{\overline{x},x}\left(0^+\right)) + \lambda\omega(\lambda),$$

where $\lambda \in [0, a(x, \overline{x})]$ and $\omega : [0, a(x, \overline{x})] \to \mathbf{R}$ satisfies $\lim_{\lambda \to 0^+} \omega(\lambda) = 0$.

In order to prove the sufficient optimality conditions we introduce the following notion.

Definition 2.4 *Let $f : X^0 \longrightarrow \mathbf{R}$ be a function, where $X^0 \subseteq \mathbf{R^n}$ is a locally arcwise connected set at $\overline{x} \in X^0$ with the corresponding function $H_{\overline{x},x}(\lambda)$ and a maximum positive number $a(x, \overline{x})$ satisfying the required conditions (from Definition 2.1). We say that f is locally PQ-connected at $\overline{x}$ $(\mathrm{LPQCN}(\overline{x}))$ if for any $x \in X^0$ there exist a positive number $d(x, \overline{x}) \leqq a(x, \overline{x})$ and an arc $H_{\overline{x},x}$ in X^0 on $[0, d(x, \overline{x})]$ such that*

$$\left.\begin{array}{c} f(x) = f(\overline{x}) \\ 0 < \lambda < d(x, \overline{x}) \end{array}\right\} \Rightarrow f\left(H_{\overline{x},x}(\lambda)\right) - f(\overline{x}) \leqq 0$$

The following results can be obtained from the above definitions.

Theorem 2.5 *Let $f : X^0 \longrightarrow \mathbf{R}$, $\overline{x} \in X^0$ where X^0 is a locally arcwise connected set at $\overline{x}$. We assume that for every $x \in X^0$, f posseses a right derivative with respect to the arc $H_{\overline{x},x}(\lambda)$ at $\lambda = 0$. Then*

a) *If the function f is ρ-$\mathrm{LCN}(\overline{x})$, then*

$$f(x) - f(\overline{x}) \geqq (\mathrm{d}f)^+(\overline{x}, H_{\overline{x},x}\left(0^+\right)) + \rho d(\overline{x}, x), \ \forall\, x \in X^0,$$

b) *If the function f is ρ-$\mathrm{LSCN}(\overline{x})$,then*

$$f(x) - f(\overline{x}) > (\mathrm{d}f)^+(\overline{x}, H_{\overline{x},x}\left(0^+\right)) + \rho d(\overline{x}, x), \ \forall\, x \in X^0, x \neq \overline{x},$$

c) *If* f *is* ρ-LQCN $(\overline{x})$, *then*

$$f(x) \leqq f(\overline{x}) \Rightarrow (\mathrm{d}f)^{+}\left(\overline{x}, H_{\overline{x},x}\left(0^{+}\right)\right) \leqq -\rho d\left(\overline{x}, x\right), x \in X^{0},$$

d) *If* f *is* ρ-LPCN $(\overline{x})$, *then*

$$(\mathrm{d}f)^{+}\left(\overline{x}, H_{\overline{x},x}\left(0^{+}\right)\right) \geqq -\rho d\left(\overline{x}, x\right) \Rightarrow f\left(x\right) \geqq f\left(\overline{x}\right), x \in X^{0},$$

e) *If* f *is* ρ-LSTPCN $(\overline{x})$, *then*

$$(\mathrm{d}f)^{+}\left(\overline{x}, H_{\overline{x},x}\left(0^{+}\right)\right) \geqq -\rho d\left(\overline{x}, x\right) \Rightarrow f\left(x\right) > f\left(\overline{x}\right), x \in X^{0}.$$

f) *If* f *is* LPQCN $(\overline{x})$, *then*

$$f\left(x\right) = f\left(\overline{x}\right) \Rightarrow (\mathrm{d}f)^{+}\left(\overline{x}, H_{\overline{x},x}\left(0^{+}\right)\right) \leqq 0.$$

3 Sufficient Optimality Criteria

In this section we will prove the sufficient conditions for optimality for Problem P. These conditions can be obtained by replacing the equality constraint $h\left(x\right) = 0$ by two inequality constraints, viz. $h\left(x\right) \leqq 0, -h\left(x\right) \leqq 0$ and then applying the results of [15]. However, in this paper we will use a direct method.

Consider the nonlinear programming problem with mixt constraints

$$\text{Min } f\left(x\right) \quad (3.1)$$

(P) subject to

$$g\left(x\right) \leqq 0,\ h\left(x\right) = 0, \quad x \in X^{0} \quad (3.2)$$

where

i) $X^{0} \subseteq \mathbf{R^{n}}$ is a nonempty open locally arcwise connected set,

ii) $f : X^{0} \rightarrow \mathbf{R}$,

iii) $g = (g_i)_{1 \leqq i \leqq m} : X^0 \to \mathbf{R}^m$

iv) $h = (h_j)_{1 \leqq j \leqq k} : X^0 \to \mathbf{R}^k$

v) the right differentials of $f, g_i(\ i = 1, \ldots, m)$ and $h_j\,(j = 1, ..., k)$ at $\overline{x}$ exist with respect to the same arc $H_{\overline{x},x}(\lambda)$.

Let $X = \{x \in X^0 \mid g(x) \leqq 0, h(x) = 0\}$ be the set of all feasible solutions to problem (P).

Let

$$N_\varepsilon(\overline{x}) = \{x \in \mathbf{R}^n \mid \|x - \overline{x}\| < \varepsilon\}.$$

Definition 3.1 a) *$\overline{x}$ is said to be a local minimum solution to problem* (P) *if $\overline{x} \in X$ and there exists $\varepsilon > 0$ such that $x \in N_\varepsilon(\overline{x}) \cap X \Rightarrow f(\overline{x}) \leqq f(x)$.*

b) *$\overline{x}$ is said to be the minimum solution to problem* (P) *if $\overline{x} \in X$ and $f(\overline{x}) = \min\limits_{x \in X} f(x)$.*

For $\overline{x} \in X$ we denote $I = I(\overline{x}) = \{i | g_i(\overline{x}) = 0\}$,(the set of indices of active constraints at $\overline{x}$), $J = J(\overline{x}) = \{i | g_i(\overline{x}) < 0\}$ (the set of indices of nonactive constraints at $\overline{x}$) and $g_I = (g_i)_{i \in I}$. Obviously $I \cup J = \{1, 2, \ldots, m\}$.

Let $u \in \mathbf{R}^m$ be such that $u \geqq 0$ and $u^T g(\overline{x}) = 0$. Evidently, $u_I \geqq 0$ and $u_J = 0$ where u_I and u_J denotes the subvectors of u corresponding to the index sets I and J, respectively.

Let $K = \{i \in I : u_i > 0\}$ and $L = \{i \in I : u_i = 0\}$; $K \cup L = I$.

Let g_K and g_L be the subvectors of g_I corresponding to the index sets K and L, respectively.

In this section we give sufficient optimality theorems of the Kuhn-Tucker and Fritz John type for problem (P).

First we give a sufficient optimality theorem of the Kuhn-Tucker type. The functions f, g and h are not differentiable but are directional differentiable with respect to the same arc $H_{\overline{x},x}(\lambda)$ at $\lambda = 0$.

The next theorem does not require the function h to be directionally differentiable.

Let $\{K_1, K_2, K_3\}$ be a partition of the index set K; thus $K_i \subset K$ for each $i = 1, 2, 3, K_r \cap K_s = \emptyset$ for each $r, s \in \{1, 2, 3\}$ with $r \neq s$, and $\overset{3}{\underset{i=1}{\cup}} K_i = K$.

Theorem 3.2 *Let $\overline{x} \in X^0 \subseteq \mathbf{R}^n$, where X^0 is a locally arcwise connected set and let $\overline{u} \in \mathbf{R}^m$. We assume that there exist the right differentials at $\overline{x}$, with respect to the same arc $H_{\overline{x},x}$ of f and g and $(\overline{x}, \overline{u})$ satisfies the following conditions:*

$$(df)^+(\overline{x}, H_{\overline{x},x}\left(0^+\right)) + \overline{u}^T(dg)^+(\overline{x}, H_{\overline{x},x}\left(0^+\right)) \geqq 0,\ \forall\, x \in X,\ (3.3)$$

$$\overline{u}^T g\left(\overline{x}\right) = 0; (3.4)$$

$$g\left(\overline{x}\right) \leqq 0; h\left(\overline{x}\right) = 0.(3.5)$$

$$\overline{u} \geqq 0, \overline{u} \neq 0(3.6)$$

Assume furthermore that the following hypotheses are satisfied:

$$\text{i}_1)\ g_i, i \in K_1, \text{is } \alpha_i\text{-LQCN}\left(\overline{x}\right), (3.7)$$

$$\text{i}_2)\ \overline{u}_{K_2}^T g_{K_2} \text{ is } \beta\text{-LQCN}\left(\overline{x}\right)(3.8)$$

$$\text{i}_3)\ f + \overline{u}_{K_3}^T g_{K_3} \text{ is } \gamma\text{-LPCN}\left(\overline{x}\right)(3.9)$$

and

$$\text{i}_4)\ \sum_{i \in K_1} \alpha_i \overline{u}_i + \beta + \gamma \geq 0.(3.10)$$

Then $\overline{x}$ is a minimum solution to Problem (P).

Proof. Let $x \in X$ be any feasible solution to problem (P). Since $g_{K_1}\left(x\right) \leqq 0 = g_{K_1}\left(\overline{x}\right)$, from (3.7), $u_{K_1} > 0$, and Theorem 2.5 we obtain the following inequality

$$\overline{u}_{K_1}^T(\mathrm{d}g_{K_1})^+(\overline{x}, H_{\overline{x},x}(0^+)) \leqq -\sum_{i \in K_1} \alpha_i \overline{u}_i d\left(x, \overline{x}\right).(3.11)$$

Also, we have

$$\overline{u}_{K_2}^T g_{K_2}(x) \leqq 0 = \overline{u}_{K_2}^T g_{K_2}(\overline{x})$$

which by (3.8) and Theorem 2.5 implies that

$$\overline{u}_{K_2}^T (\mathrm{d}g_{K_2})^+ (\overline{x}, H_{\overline{x},x}(0^+)) \leqq -\beta d(x, \overline{x}) . (3.12)$$

Since $\overline{u}_J = 0$ and $\overline{u}_L = 0$, it follows from (3.3) that

$$(\mathrm{d}f)^+(\overline{x}, H_{\overline{x},x}(0^+)) + \overline{u}_{K_3}^T (\mathrm{d}g_{K_3})^+(\overline{x}, H_{\overline{x},x}(0^+)) \geqq$$

$$-\overline{u}_{K_1}^T (\mathrm{d}g_{K_1})^+(\overline{x}, H_{\overline{x},x}(0^+)) - \overline{u}_{K_2}^T (\mathrm{d}g_{K_2})^+(\overline{x}, H_{\overline{x},x}(0^+)) . (3.13)$$

From (3.11), (3.12) and (3.13) we see that

$$(\mathrm{d}f)^+(\overline{x}, H_{\overline{x},x}(0^+)) + \overline{u}_{K_3}^T (\mathrm{d}g_{K_3})^+(\overline{x}, H_{\overline{x},x}(0^+)) \geqq \left(\sum_{i \in K_1} \alpha_i \overline{u}_i + \beta \right) d(x, \overline{x})$$

which in view of (3.10) implies that

$$(\mathrm{d}f)^+(\overline{x}, H_{\overline{x},x}(0^+)) + \overline{u}_{K_3}^T (\mathrm{d}g_{K_3})^+(\overline{x}, H_{\overline{x},x}(0^+)) \geqq -\gamma d(x, \overline{x})$$

or

$$\left(\mathrm{d}\left(f + \overline{u}_{K_3}^T g_{K_3}\right)\right)^+ \left(\overline{x}, H_{\overline{x},x}(0^+)\right) \geqq -\gamma \mathrm{d}(x, \overline{x})$$

From (3.9) and Theorem 2.5 we deduce that

$$f(x) + \overline{u}_{K_3}^T g_{K_3}(x) \geqq f(\overline{x}) + \overline{u}_{K_2}^T g_{K_2}(\overline{x}) . (3.14)$$

Inasmuch $\overline{u}_{K_3}^T g_{K_3}(x) \leqq 0$ and $\overline{u}_{K_3}^T g_{K_3}(\overline{x}) = 0$, it follows from (3.14) that $f(x) \geqq f(\overline{x})$. Hence $\overline{x}$ is a minimum solution to problem (P).

The above theorem has a number of special cases which easily can be identified by suitable choices of the partitioning sets $\{K_1, K_2, K_3\}$. We shall state these cases as an corollary.

Corollary 3.3 *Let $\overline{x} \in X^0 \subseteq \mathbf{R}^n$, where X^0 is a locally arcwise connected set and let $\overline{u} \in \mathbf{R}^m$. We assume that there exist the right differential at $\overline{x}$ with respect to the same arc $H_{\overline{x},x}$ of f and g and $(\overline{x},\overline{u})$ satisfies conditions* (3.3) – (3.6).

Assume furthermore that any one of the following hypotheses is satisfied.

i$_1$) $f + \overline{u}_K^T g_K$ *is* γ*-LPCN*$(\overline{x})$, *where* $\gamma \geqq 0$,

i$_2$) a) $g_i, i \in K$, *is* α_i*-LQCN*$(\overline{x})$

b) f *is* γ*-LPCN*$(\overline{x})$

c) $_{i\in K}\alpha_i\overline{u}_i + \gamma \geq 0.$

i$_3$) a) $\overline{u}_K^T g_K$ *is* β*-LQCN*$(\overline{x})$,

b) f *is* γ*-LPCN*$(\overline{x})$,

c) $\beta + \gamma \geq 0$,

i$_4$) a) $\overline{u}_{K_2}^T g_{K_2}$ *is* β*-LQCN*$(\overline{x})$,

b) $f + \overline{u}_{K_3}^T g_{K_3}$ *is* γ*-LPCN*$(\overline{x})$, *where* $\{K_2, K_3\}$ *is a partition of* K,

c) $\beta + \gamma \geq 0.$

i$_5$) a) $g_i, i \in K_1$, *is* α_i*-LQCN*$(\overline{x})$,

b) $f + \overline{u}_{K_3}^T g_{K_3}$ *is* γ*-LPCN*$(\overline{x})$, *where* $\{K_1, K_3\}$ *is a partition of* K,

c)$_{i\in K_1}\alpha_i\overline{u}_i + \gamma \geq 0$,

i$_6$) a) $g_i, i \in K_1$, *is* α_i*-LQCN*$(\overline{x})$,

b) $\overline{u}_{K_2}^T g_{K_2}$ *is* β*-LQCN*$(\overline{x})$,

c) f *is* γ*-LPCN*$(\overline{x})$

d) $_{i\in K_1}\alpha_i\overline{u}_i + \beta + \gamma \geq 0$, *where* $\{K_1, K_2\}$ *is a partition of* K.

Then $\overline{x}$ is a minimum solution to problem (P).

Proof. Each of the six sets of conditions given in Corollary 3.3. can be considered as a family of sufficient optimality conditions whose members can easily be identified by appropriate choices of the partitioning sets $\{K_1, K_2, K_3\}$. In Theorem 3.2, let i$_1$) $K_1 = K_2 = \emptyset, K_3 = K$, i$_2$) $K_1 = K, K_2 = K_3 = \emptyset$, i$_3$) $K_1 = K_3 = \emptyset, K_2 = K$, i$_4$) $K_1 = \emptyset, K_2 \neq \emptyset, K_3 \neq \emptyset$, i$_5$) $K_1 \neq \emptyset, K_2 = \emptyset, K_3 \neq \emptyset$, and i$_6$) $K_1 \neq \emptyset, K_2 \neq \emptyset, K_3 = \emptyset$.

Let $v \in \mathbf{R}^k$ and define $P = \{i | v_i > 0\}$ and $Q = \{i | v_i < 0\}$. Let $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ be partitions of the sets P and Q, respectively.

Let h_{P_i} and h_{Q_i} $(i = 1, 2, 3)$ be the subvectors of h corresponding to the index sets P_i and Q_i $(i = 1, 2, 3)$, respectively. Let v_{P_i} and v_{Q_i} $(i = 1, 2, 3)$ be the subvectors of v to the index sets P_i and Q_i $(i = 1, 2, 3)$, respectively.

The next theorem does not require the function g to be directionally differentiable.

Theorem 3.4 *Let $\overline{x} \in X^0 \subseteq \mathbf{R}^n$, where X^0 is a locally arcwise connected set and let $\overline{v} \in \mathbf{R}^k$. We assume that there exist the right differentials at $\overline{x}$, with respect to the same arc $H_{\overline{x},x}$ of f and h and $(\overline{x}, \overline{v})$ satisfies the following condition:*

$$(\mathrm{d}f)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) + \overline{v}^T (\mathrm{d}h)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) \geqq 0, \ \forall x \in X. (3.15)$$

Assume furthermore that

$$\mathrm{i}_1) \ h_i, \ i \in P_1, \ is \ \mathrm{LPQCN}\left(\overline{x}\right), (3.16)$$

$$\mathrm{i}_2) \ -h_i, \ i \in Q_1, is \ \mathrm{LPQCN}\left(\overline{x}\right), (3.17)$$

$$\mathrm{i}_3) \ \overline{v}_{P_2}^T h_{P_2} + \overline{v}_{Q_2}^T h_{Q_2} is \ \mathrm{LPQCN}\left(\overline{x}\right), (3.18)$$

and

$$\mathrm{i}_4) \ f + v_{P_3}^T h_{P_3} + v_{Q_3}^T h_{Q_4} \ is \ \tau - \mathrm{LPCN}\left(\overline{x}\right), \ (\tau \geq 0) \ (3.19)$$

Then $\overline{x}$ is a minimum solution to Problem (P).

Proof. Let $x \in X$ be any feasible solution to problem (P). Since $h_i(x) = h_i(\overline{x}), i \in P_1$, from (3.16) and Theorem 2.5 we obtain

$$(\mathrm{d}h_i)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) \leqq 0$$

for any $i \in P_1$.

Now multiplying this inequality by $v_i, i \in P_1$, and summing, we obtain

$$\overline{v}_{P_1}^T \left(\mathrm{d}h_{P_1}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) \leqq 0.(3.20)$$

Also, we have

$$h_i\left(x\right) = h_i\left(\overline{x}\right), \ i \in Q_1$$

which by (3.17) together with $v_{Q_1} < 0$ and Theorem 2.5 implies that

$$\overline{v}_{Q_1}^T \left(\mathrm{d}h_{Q_1}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) \leqq 0.(3.21)$$

We also have

$$\overline{v}_{P2}^T h_{P_2}\left(x\right) + \overline{v}_{Q_2}^T h_{Q_2}\left(x\right) = \overline{v}_{P_2}^T h_{P_2}\left(\overline{x}\right) + \overline{v}_{Q_2}^T h_{Q_2}\left(\overline{x}\right)$$

which by (3.18) and Theorem 2.5 implies that

$$\overline{v}_{P_2}^T \left(\mathrm{d}h_{P_2}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) + \overline{v}_{Q_2}^T \left(\mathrm{d}h_{Q_2}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) \leqq 0.(3.22)$$

The relation (3.15) can be written as

$$(\mathrm{d}f)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) + \overline{v}_{P_1}^T \left(\mathrm{d}h_{P_1}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) + \overline{v}_{P_2}^T \left(\mathrm{d}h_{P_2}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) +$$

$$\overline{v}_{P_3}^T \left(\mathrm{d}h_{P_3}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) + \overline{v}_{Q_1}^T \left(\mathrm{d}h_{Q_1}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) +$$

$$+\overline{v}_{Q_2}^T \left(\mathrm{d}h_{P_2}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) + \overline{v}_{Q_3}^T \left(\mathrm{d}h_{Q_3}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) \geqq 0, \ \forall \ x \in X$$

which in view of (3.20), (3.21) and (3.22) implies that

$$(\mathrm{d}f)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) + \overline{v}_{P_3}^T \left(\mathrm{d}h_{P_3}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) + \overline{v}_{Q_3}^T \left(\mathrm{d}h_{Q_3}\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) \geqq 0$$

or

$$\left(\mathrm{d}\left(f + \overline{v}_{P_3}^T h_{P_3} + \overline{v}_{Q_3}^T h_{Q_3}\right)\right)^+ \left(\overline{x}, H_{\overline{x},x}\left(0^+\right)\right) \geqq 0 \geqq -\tau d\left(x, \overline{x}\right).$$

Using (3.19) and Theorem 2.5 we deduce that

$$f(x) + \overline{v}_{P_3}^T h_{P_3}(x) + \overline{v}_{Q_3}^T h_{Q_3}(x) \geqq f(\overline{x}) + \overline{v}_{P_3}^T h_{P_3}(\overline{x}) + \overline{v}_{Q_3}^T h_{Q_3}(\overline{x})$$

which implies

$$f(x) \geqq f(\overline{x}).$$

Hence $\overline{x}$ is a minimum solution to Problem (P_1).

Let $u \in \mathbf{R}^m$. Let $L = \{i | u_i > 0\}$. Let $v \in \mathbf{R}^k$ and define $P = \{i | v_i > 0\}$ and $Q = \{i | v_i < 0\}$. Let $\{L_1, L_2, L_3, L_4\}, \{P_1, P_2, P_3, P_4\}$ and $\{Q_1, Q_2, Q_3, Q_4\}$ be partitions of the sets L, P and Q, respectively.

The following theorem is a combination of Theorems 3.4 and 3.5.

Theorem 3.5 *Let $\overline{x} \in X^0 \subseteq \mathbf{R}^n$, where X^0 is a locally arcwise connected set and let $\overline{u} \in \mathbf{R}^m$ and $\overline{v} \in \mathbf{R}^k$. We assume that there exist the right differentials at $\overline{x}$, with respect to the same arc $H_{\overline{x},x}$ of f, g and h and $(\overline{x}, \overline{u}, \overline{v})$ satisfies the following conditions:*

$$(\mathrm{d}f)^+ (\overline{x}, H_{\overline{x},x}(0^+)) + \overline{u}^T (\mathrm{d}g)^+ (\overline{x}, H_{\overline{x},x}(0^+)) + \overline{v} (\mathrm{d}h)^+ (\overline{x}, H_{\overline{x},x}(0^+)) \geqq 0, \ \forall x \in X$$

$$\overline{u}^T g(\overline{x}) = 0$$

$$g(\overline{x}) \leqq 0, h(\overline{x}) = 0$$

$$\overline{u} \geqq 0$$

Assume furthermore that

i_1) $g_i,\ i \in L_1,$ *is* $\alpha_i - \mathrm{LQCN}(\overline{x})$,

i_2) $h_i,\ i \in P_1,$ *is LPQCN*$(\overline{x})$,

i_3) $-h_i,\ i \in Q_1,$ *is LPQCN*$(\overline{x})$,

i_4) $\overline{u}_{L_2}^T g_{L_2}$ *is* β*-LQCN*$(\overline{x})$,

i_5) $\overline{v}_{P_2}^T h_{P_2} + \overline{v}_{Q_2}^T h_{Q_2}$ *is* $\mathrm{LPQCN}(\overline{x})$,

i_6) $\overline{u}_{L_3}^T g_{L_3} + \overline{v}_{P_3}^T h_{P_3} + \overline{v}_{Q_3}^T h_{Q_3}$ *is* $\delta - \mathrm{LQCN}(\overline{x})$,

i_7) $f+\overline{u}_{L_4}^T g_{L_4}+\overline{v}_{P_4}^T h_{P_4}+\overline{v}_{Q_4}^T h_{Q_4}$ *is* $\tau-$ LPQCN $(\overline{x})$,

i_8) $\sum_{i\in L_1}\overline{u}_i\alpha_i+\beta+\delta+\tau\geq 0.$

Then $\overline{x}$ *is a minimum solution to Problem* (P).

The proof of this theorem is similar to that of Theorems 3.2 and 3.4.

In what follows we consider sufficient optimality conditions of the Fritz John type.

Let $(\overline{x}, v_0, v, w)$ be a Fritz John point, where $\overline{x}\in X^0$(a locally arcwise connected set), $v_0\in\mathbf{R}$, $v\in\mathbf{R}^m$ and $w\in\mathbf{R}^k$. Assume that $(\overline{x}, v_0, v, w)$ satisfies the following conditions:

$$v_0(\mathrm{d}f)^+(\overline{x},H_{\overline{x},x}\left(0^+\right))+v^T(\mathrm{d}g)^+(\overline{x},H_{\overline{x},x}\left(0^+\right))+w^T(\mathrm{d}h)^+(\overline{x},H_{\overline{x},x}\left(0^+\right))\geqq 0,\ \forall\, x\in X \tag{3.23}$$

$$v^T g\left(\overline{x}\right)=0 \tag{3.24}$$

$$(v_0, v)\geqq 0, (v_0, v, w)\neq 0. \tag{3.25}$$

If $v_0=0$, then conditions (3.23)-(3.25) become

$$v^T(\mathrm{d}g)^+(\overline{x},H_{\overline{x},x}\left(0^+\right))+w^T(\mathrm{d}h)^+(\overline{x},H_{\overline{x},x}\left(0^+\right))\geqq 0,\ \forall\, x\in X \tag{3.26}$$

$$v^T g\left(\overline{x}\right)=0 \tag{3.27}$$

$$v\geqq 0,\ (v,w)\neq 0. \tag{3.28}$$

Let I and J be the sets defined at the beginning of this section. Due to the relation (3.24) we have $v_I\geqq 0$ and $v_J=0$. Let $L=\{i\in I: v_i>0\}$. Let g_L be the subvector of g_I corresponding to the index set L. Also, let v_L be the subvector of v corresponding to the index set L.

Let $w \in \mathbf{R}^k$. Define the index sets U and V by $U = \{i | w_i > 0\}$ and $V = \{i | w_i < 0\}$. Let h_U and h_V be the subvectors of h corresponding to the index sets U and V, respectively. Also, let w_U and w_V be the vectors of W corresponding to the index sets U and V, respectively.

Theorem 3.6 *Let $\overline{x} \in X^0 \subseteq \mathbf{R}^n$, where X^0 is a locally arcwise connected set. We assume that there exist the right differential at $\overline{x}$, with respect to the same arc $H_{\overline{x},x}$ of f, g and h. Let $(\overline{x}, v_0, v, w)$ be a Fritz John point which satisfy conditions* $(3.23) - (3.25)$.

i) *If $v_0 > 0$, let the assumptions of Theorem 3.5 hold with*

$$\overline{u} = v_0^{-1} v \text{ and } \overline{v} = v_0^{-1} w$$

ii) *If $v_0 = 0$, let $(\overline{x}, 0, v, w)$ satisfy* $(3.26) - (3.28)$ *and the following hypotheses are satisfied*

a) $g_i, i \in L_1,$ is α_i-LQCN $(\overline{x})$,

b) $h_i, i \in U_1,$ is $LPQCN$ $(\overline{x})$,

c) $-h_i, i \in V_1,$ is $LPQCN$ $(\overline{x})$,

d) $v_{L_2}^T g_{L_2} is$ β-LQCN $(\overline{x})$,

e) $w_{U_2}^T h_{U_2} + w_{V_2}^T h_{V_2}$ is LPQCN $(\overline{x})$

f) $v_{L_3}^T g_{L_3} + w_{U_3}^T h_{U_3} + w_{V_3}^T h_{V_3}$ is $\delta -$ LQCN $(\overline{x})$

g) $_{i \in L_1} \alpha_i v_i + \beta + \delta > 0,$

Then $\overline{x}$ is a global minimum solution to Problem (P).

The proof of this theorem is similar to that of Theorem 3.4 from [15]. Hence the proof is submitted.

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