NONDIFFERENTIABLE MINIMAX FRACTIONAL PROGRAMMING WITH SQUARE ROOT TERMS

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Abstract

We establish necessary and sufficient optimality condition for a class of nondifferentiable minimax fractional programming problems with square root terms involving (η, ρ, θ)-invex funtions. Subsequently, we apply the optimality condition to formulate a parametric dual problem and we prove weak duality, strong duality, and strict converse duality theorems.

1. Introduction

Let us consider the following continuous differentiable mappings:

$$\begin{aligned} f &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \quad h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \\ g &: \mathbb{R}^n \to \mathbb{R}^p, \end{aligned}$$

with $g = (g_1, \cdots, g_p)$. We denote

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid g_j(x) \leq 0,\ j = 1, 2, \cdots, p\} \tag{1.1}$$

and consider $Y \subseteq \mathbb{R}^m$ to be a compact subset of $\mathbb{R}^m$. Let B_r, $r = \overline{1, \beta}$, and D_q, $q = \overline{1, \delta}$, be $n \times n$ positive semidefinite matrices such that for each $(x, y) \in \mathcal{P} \times Y$, we have:

$$\begin{aligned} f(x, y) + \sum_{r=1}^{\beta} \sqrt{x^\top B_r x} &\geq 0 \\ h(x, y) - \sum_{q=1}^{\delta} \sqrt{x^\top D_q x} &> 0 \end{aligned}$$

In this paper we consider the following nondifferentiable minimax fractional programming problem:

$$\inf_{x\in\mathcal{P}} \sup_{y\in Y} \frac{f(x,y)+\sum_{r=1}^{\beta}\sqrt{x^{\top}B_r x}}{h(x,y)-\sum_{q=1}^{\delta}\sqrt{x^{\top}D_q x}} \tag{P}$$

For $\beta = \delta = 1$, this problem was studied by Lai et al. [14], and further, if $B_1 = D_1 = 0$, (P) is a differentiable minimax fractional programming problem which has been studied by Chandra and Kumar [7], Liu and Wu [16]. Many authors investigated the optimality conditions and duality theorems for minimax (fractional) programming problems. For details, one can consult [1, 4, 14, 15, 21, 26]. Problems which contain square root terms were first studied by Mond [18]. Some extensions of Mond's results were obtained, for example, by Chandra et al. [5], Preda [20], Zhang and Mond [30], Preda and Köller [22].

In an earlier work, under conditions of convexity, Schmittendorf [25] established necessary and sufficient optimality conditions for the problem:

$$\inf_{x\in\mathcal{P}} \sup_{y\in Y} \psi(x,y), \tag{P1}$$

where $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a continuous differentiable mapping. Later, Yadev and Mukherjee [28] employed the optimality conditions of Schmittendorf [25] to construct two dual problems and they derived duality theorems for (convex) differentiable fractional minimax programming. In [7], Chandra and Kumar constructed two modified dual problems for which they proved duality theorems for (convex) differentiable fractional minimax programming. Liu and Wu [16] relaxed the convexity assumption in the sufficient optimality of [7] and they employed the optimality conditions to construct one parametric dual and two other dual models of parametric-free problems, and they established weak duality, strong duality, and strict converse duality theorems for a class of generalized minimax fractional programming involving generalized convex functions. Several authors considered the optimality and duality theorems for nondifferentiable nonconvex minimax fractional programming problems, one can consult [15, 26, 29].

In this paper, we present necessary and sufficient optimality conditions for problem (P) and we apply the optimality conditions to construct one parametric dual problem. Further, we establish for this pair of dual problems weak duality, strong duality, and strictly converse duality theorems. Some definitions and notations are given in Section 2. In Section 3, necessary optimality contitions are proved and we derive sufficient conditions under the assumption of generalized convexity. Using the optimality conditions, in Section 4 we state the parametric dual problem and prove the duality results.

2. Notations and Preliminary Results

Throughout this paper, we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by $\mathbb{R}^n_+$ its nonnegative orthant. Let us consider the set $\mathcal{P}$ defined by (1.1), and for each $x \in \mathcal{P}$, we define

$$\begin{aligned} J(x) &= \{j \in \{1,2,\cdots,p\} \mid g_j(x) = 0\}, \\ Y(x) &= \left\{ y \in Y \;\middle|\; \frac{f(x,y)+\sum_{r=1}^{\beta}\sqrt{x^\top B_r x}}{h(x,y)-\sum_{q=1}^{\delta}\sqrt{x^\top D_q x}} = \sup_{z\in Y} \frac{f(x,z)+\sum_{r=1}^{\beta}\sqrt{x^\top B_r x}}{h(x,z)-\sum_{q=1}^{\delta}\sqrt{x^\top D_q x}} \right\}, \\ K(x) &= \left\{ (s,t,\bar{y}) \in \mathbb{N}\times\mathbb{R}^s_+\times\mathbb{R}^{ms} \;\middle|\; \begin{array}{l} 1 \le s \le n+1,\ \sum_{i=1}^{s} t_i = 1, \\ \text{and } \bar{y} = (\bar{y}_1,\cdots,\bar{y}_s) \in \mathbb{R}^{ms} \\ \text{with } \bar{y}_i \in Y(x),\ i = \overline{1,s} \end{array} \right\}. \end{aligned}$$

Since f and h are continuous differentiable functions and Y is a compact set in $\mathbb{R}^m$, it follows that for each $x_0 \in \mathcal{P}$, we have $Y(x_0) \neq \emptyset$, and for any $\bar{y}_i \in Y(x_0)$, we denote

$$k_0 = \frac{f(x_0,\bar{y}_i)+\sum_{r=1}^{\beta}\sqrt{x_0^\top B_r x_0}}{h(x_0,\bar{y}_i)-\sum_{q=1}^{\delta}\sqrt{x_0^\top D_q x_0}}. \tag{2.1}$$

Let A be an $m \times n$ matrix and let M, M_i, $i = 1,\cdots,k$, be $n \times n$ symmetric positive semidefinite matrices.

Lemma 2.1 [27] *We have*

$$Ax \ge 0 \ \Rightarrow\ c^\top x + \sum_{i=1}^{k}\sqrt{x^\top M_i x} \ge 0,$$

if and only if there exist $y \in \mathbb{R}^m_+$ and $v_i \in \mathbb{R}^n$, $i = \overline{1,k}$, such that

$$Av_i \ge 0, \quad v_i^\top M_i v_i \le 1,\ i = \overline{1,k}, \quad A^\top y = c + \sum_{i=1}^{k} M_i v_i.$$

If all $M_i = 0$, Lemma 2.1 becomes the well-known Farkas lemma.

We shall use the generalized Schwarz inequality [23]:

$$x^\top M v \le \sqrt{x^\top M x}\sqrt{v^\top M v}. \tag{2.2}$$

We note that equality holds in (2.2) if $Mx = \tau M v$ for some $\tau \ge 0$.

Obviously, from (2.2), we have

$$v^\top Mv \le 1 \;\Rightarrow\; x^\top Mv \le \sqrt{x^\top Mx}. \tag{2.3}$$

The following lemma is given by Schmittendorf [25] for the problem (P1):

Lemma 2.2 [25] *Let x_0 be a solution of the minimax problem (P1) and the vectors $\nabla g_j(x_0)$, $j \in J(x_0)$ are linearly independent. Then there exist a positive integer s, $1 \le s \le n+1$, real numbers $t_i \ge 0$, $i = \overline{1,s}$, $\mu_j \ge 0$, $j = \overline{1,p}$, and vectors $\bar{y}_i \in Y(x_0)$, $i = \overline{1,s}$, such that*

$$\sum_{i=1}^{s} t_i \nabla_x \psi(x_0, \bar{y}_i) + \sum_{j=1}^{p} \mu_j \nabla g_j(x_0) = 0,$$
$$\mu_j g_j(x_0) = 0, \quad j = \overline{1,p},$$
$$\sum_{i=1}^{s} t_i \ne 0.$$

Now we give the definitions of (η, ρ, θ)-quasi-invexity and (η, ρ, θ)-pseudo-invexity as extensions of the invexity notion. The invexity notion of a function was introduced into optimization theory by Hanson [11] and the name of invex function was given by Craven [8]. Some extensions of invexity as pseudo-invexity, quasi-invexity and ρ-invexity, ρ-pseudo-invexity, ρ-quasi-invexity are presented, for example, in Craven and Glover [9], Kaul and Kaur [13], Preda [19], Mititelu and Stancu-Minasian [17]. In this paper we only shall use the following notions:

Definition 2.1 A differentiable function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is (η, ρ, θ)-invex at $x_0 \in C$ if there exist functions $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ and $\rho \in \mathbb{R}$ such that

$$\varphi(x) - \varphi(x_0) \ge \eta(x, x_0)^\top \nabla\varphi(x_0) + \rho\theta(x, x_0).$$

If $-\varphi$ is (η, ρ, θ)-invex at $x_0 \in C$, then φ is called (η, ρ, θ)-incave at $x_0 \in C$.
If the inequality holds strictly, then φ is called to be strictly (η, ρ, θ)-invex.

Definition 2.2 A differentiable function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is (η, ρ, θ)-pseudo-invex at $x_0 \in C$ if there exist functions $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ and $\rho \in \mathbb{R}$ such that the following hold:

$$\eta(x, x_0)^\top \nabla\varphi(x_0) \ge -\rho\theta(x, x_0) \implies \varphi(x) \ge \varphi(x_0), \quad \forall x \in C,$$

or equivalently,

$$\varphi(x) < \varphi(x_0) \implies \eta(x, x_0)^\top \nabla\varphi(x_0) < -\rho\theta(x, x_0), \quad \forall x \in C.$$

If $-\varphi$ is (η, ρ, θ)-pseudo-invex at $x_0 \in C$, then φ is called (η, ρ, θ)-pseudo-incave at $x_0 \in C$.

Definition 2.3 A differentiable function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is strictly (η, ρ, θ)-pseudo-invex at $x_0 \in C$ if there exist functions $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ and $\rho \in \mathbb{R}$ such that the following hold:

$$\eta(x, x_0)^\top \nabla\varphi(x_0) \geq -\rho\theta(x, x_0) \implies \varphi(x) > \varphi(x_0), \quad \forall\, x \in C,\ x \neq x_0.$$

Definition 2.4 A differentiable function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is (η, ρ, θ)-quasi-invex at $x_0 \in C$ if there exist functions $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ and $\rho \in \mathbb{R}$ such that the following hold:

$$\varphi(x) \leq \varphi(x_0) \implies \eta(x, x_0)^\top \nabla\varphi(x_0) \leq -\rho\theta(x, x_0), \quad \forall\, x \in C.$$

If $-\varphi$ is (η, ρ, θ)-quasi-invex at $x_0 \in C$, then φ is called (η, ρ, θ)-quasi-incave at $x_0 \in C$.

If in the above definitions the corresponding property of a differentiable function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is satisfied for any $x_0 \in C$, then we say that φ has the respective (η, ρ, θ)-characteristic on C.

3. Necessary and Sufficient Optimality Conditions

For any $x \in \mathcal{P}$, let us denote the following index sets:

$$\begin{aligned}
\mathcal{B}(x) &= \left\{r \in \{1, 2, \cdots, \beta\} \mid x^\top B_r x > 0\right\}, \\
\overline{\mathcal{B}}(x) &= \{1, 2, \cdots, \beta\} \setminus \mathcal{B}(x) = \left\{r \mid x^\top B_r x = 0\right\}, \\
\\
\mathcal{D}(x) &= \left\{q \in \{1, 2, \cdots, \delta\} \mid x^\top D_q x > 0\right\}, \\
\overline{\mathcal{D}}(x) &= \{1, 2, \cdots, \delta\} \setminus \mathcal{D}(x) = \left\{q \mid x^\top D_q x = 0\right\}.
\end{aligned}$$

Using Lemma 2.2, we may prove the following necessary optimality conditions for problem (P).

Theorem 3.1 (Necessary Condition) *If x_0 is an optimal solution of problem (P) for which $\overline{\mathcal{B}}(x_0) = \emptyset$, $\overline{\mathcal{D}}(x_0) = \emptyset$, and $\nabla g_j(x_0)$, $j \in J(x_0)$ are linearly independent, then there exist $(s, \bar{t}, \bar{y}) \in K(x_0)$, $k_0 \in \mathbb{R}_+$, $w_r \in \mathbb{R}^n, r = \overline{1, \beta}$, $v_q \in \mathbb{R}^n, q = \overline{1, \delta}$, and $\bar{\mu} \in \mathbb{R}^p_+$ such that*

$$\begin{aligned}
\sum_{i=1}^{s} \bar{t}_i \left[\nabla f(x_0, \bar{y}_i) + \sum_{r=1}^{\beta} B_r w_r - k_0 \left(\nabla h(x_0, \bar{y}_i) - \sum_{q=1}^{\delta} D_q v_q\right)\right] + \\
+ \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(x_0) = 0,
\end{aligned} \tag{3.1}$$

$$f\left(x_0,\bar{y}_i\right)+\sum_{r=1}^{\beta}\sqrt{x_0^\top B_r x_0}-k_0\left(h\left(x_0,\bar{y}_i\right)-\sum_{q=1}^{\delta}\sqrt{x_0^\top D_q x_0}\right)=0,\quad \forall\, i=\overline{1,s}, \tag{3.2}$$

$$\sum_{j=1}^{p}\bar{\mu}_j g_j\left(x_0\right)=0, \tag{3.3}$$

$$\bar{t}_i\geq 0,\qquad \sum_{i=1}^{s}\bar{t}_i=1, \tag{3.4}$$

$$\left.\begin{array}{ll} w_r^\top B_r w_r\leq 1,\quad x_0^\top B_r w_r=\sqrt{x_0^\top B_r x_0}, & r=\overline{1,\beta},\\ v_q^\top D_q v_q\leq 1,\quad x_0^\top D_q v_q=\sqrt{x_0^\top D_q x_0} & q=\overline{1,\delta}. \end{array}\right\} \tag{3.5}$$

Proof. Since all $B_r, r=\overline{1,\beta}$, and $D_q, q=\overline{1,\delta}$, are positive definite and f and h are differentiable functions, it follows that the function

$$\frac{f\left(x,y\right)+\sum\limits_{r=1}^{\beta}\sqrt{x^\top B_r x}}{h\left(x,y\right)-\sum\limits_{q=1}^{\delta}\sqrt{x^\top D_q x}}$$

is differentiable with respect to x for any given $y\in\mathbb{R}^m$. In Lemma 2.2, the differentiable function ψ in (P1) is replaced by the objective (fractional) function of (P), and, like the Kuhn-Tucker type formula, it follows that there exist a positive integer s, $1\leq s\leq n+1$, and vectors $t\in\mathbb{R}_+^s$, $\bar{\mu}\in\mathbb{R}_+^p$, $\bar{y}_i\in Y\left(x_0\right)$, $i=\overline{1,s}$, such that

$$\begin{aligned}\sum_{i=1}^{s}t_i\frac{1}{h\left(x_0,\bar{y}_i\right)-\sum\limits_{q=1}^{\delta}\sqrt{x_0^\top D_q x_0}}&\left[\nabla f\left(x_0,\bar{y}_i\right)+\sum_{r=1}^{\beta}\frac{B_r x_0}{\sqrt{x_0^\top B_r x_0}}-\right.\\ &\left.-\,k_0\left(\nabla h\left(x_0,\bar{y}_i\right)-\sum_{r=1}^{\beta}\frac{D_q x_0}{\sqrt{x_0^\top D_q x_0}}\right)\right]+\sum_{j=1}^{p}\bar{\mu}_j\nabla g_j\left(x_0\right)=0\end{aligned} \tag{3.6}$$

$$\sum_{j=1}^{p}\bar{\mu}_j g_j\left(x_0\right)=0, \tag{3.7}$$

$$\sum_{i=1}^{s}t_i>0, \tag{3.8}$$

where k_0 is given by (2.1).

Now let us denote

$$\begin{aligned}
w_r &= \frac{x_0}{\sqrt{x_0^\top B_r x_0}}, \qquad r = \overline{1,\beta}, \\
v_q &= \frac{x_0}{\sqrt{x_0^\top D_q x_0}}, \qquad q = \overline{1,\delta}, \\
\bar{t}_i &= \frac{t_i^0}{\sum\limits_{i=1}^{s} t_i^0}, \qquad \text{where } t_i^0 = \frac{t_i}{h(x_0, \bar{y}_i) - \sum\limits_{q=1}^{\delta} \sqrt{x_0^\top D_q x_0}},
\end{aligned}$$

Equations (3.6) and (3.7) become

$$\sum_{i=1}^{s} \bar{t}_i \left[\nabla f(x_0, \bar{y}_i) + \sum_{r=1}^{\beta} B_r w_r - k_0 \left(\nabla h(x_0, \bar{y}_i) - \sum_{q=1}^{\delta} D_q v_q \right) \right] + \\ + \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(x_0) = 0,$$

$$\sum_{j=1}^{p} \bar{\mu}_j g_j(x_0) = 0,$$

where $\bar{\mu} \in \mathbb{R}_+^p$, $\bar{t}_i \geq 0$ for all $i = \overline{1,s}$, with $\sum\limits_{i=1}^{s} \bar{t}_i > 0$. This proves (3.1) - (3.4). Furthermore, it verifies easely that we have

$$\begin{aligned}
w_r^\top B_r w_r = 1, \quad x_0^\top B_r w_r = \sqrt{x_0^\top B_r x_0}, \quad \text{for any } r = \overline{1,\beta}, \\
v_q^\top D_q v_q = 1, \quad x_0^\top D_q v_q = \sqrt{x_0^\top D_q x_0}, \quad \text{for any } q = \overline{1,\delta}.
\end{aligned}$$

So relation (3.5) also holds, and the theorem is proved. ■

We notice that, in the above theorem, all matrices B_r and D_q are supposed to be positive definite. If at least one of $\overline{\mathcal{B}}(x_0)$ or $\overline{\mathcal{D}}(x_0)$ is not empty, then the functions involved in the objective function of problem (P) are not differentiable. In this case, the necessary optimality conditions still hold under some additional assumptions. For $x_0 \in \mathcal{P}$ and $(s, \bar{t}, \bar{y}) \in K(x_0)$ we define the following vector:

$$\alpha = \sum_{i=1}^{s} \bar{t}_i \left(\nabla f(x_0, \bar{y}_i) + \sum_{r \in \mathcal{B}(x_0)} \frac{B_r x_0}{\sqrt{x_0^\top B_r x_0}} - \right. \\ \left. - k_0 \left(\nabla h(x_0, \bar{y}_i) - \sum_{r \in \mathcal{D}(x_0)} \frac{D_q x_0}{\sqrt{x_0^\top D_q x_0}} \right) \right)$$

Now we define a set Z as follows:

$$Z_{\bar{y}}(x_0) = \left\{ z \in \mathbb{R}^n \;\middle|\; \begin{array}{c} z^\top \nabla g_j(x_0) \le 0,\ j \in J(x_0), \\ \text{and the relation (3.9) holds.} \end{array} \right\}$$

$$z^\top \alpha + \sum_{i=1}^{s} \bar{t}_i \left(\sum_{r \in \overline{\mathcal{B}}(x_0)} \sqrt{z^\top B_r z} + \sum_{q \in \overline{\mathcal{D}}(x_0)} \sqrt{z^\top \left((k_0)^2 D_q \right) z} \right) < 0. \tag{3.9}$$

If one of the index sets involved in the above expressions is empty, then the correspondig sum vanishes.

Using Lemma 2.1, we establish the following result:

Theorem 3.2 *Let x_0 be an optimal solution of problem (P) and at least one of $\overline{\mathcal{B}}(x_0)$ or $\overline{\mathcal{D}}(x_0)$ is not empty. Let $(s, \bar{t}, \bar{y}) \in K(x_0)$ be such that $Z_{\bar{y}}(x_0) = \emptyset$. Then there exist vectors $w_r \in \mathbb{R}^n$, $r = \overline{1, \beta}$, $v_q \in \mathbb{R}^n$, $q = \overline{1, \delta}$, and $\bar{\mu} \in \mathbb{R}_+^p$ which satisfy the relations (3.1) - (3.5).*

Proof. Using (2.1) we get (3.2), and relation (3.4) follows directly from the assumptions.

Since $Z_{\bar{y}}(x_0) = \emptyset$, for any $z \in \mathbb{R}^n$ with: $-z^\top \nabla g_j(x_0) \ge 0,\ j \in J(x_0)$, we have

$$z^\top \alpha + \sum_{i=1}^{s} \bar{t}_i \left(\sum_{r \in \overline{\mathcal{B}}(x_0)} \sqrt{z^\top B_r z} + \sum_{q \in \overline{\mathcal{D}}(x_0)} \sqrt{z^\top \left((k_0)^2 D_q \right) z} \right) \ge 0.$$

Let us denote:

$$\lambda = \sum_{i=1}^{s} \bar{t}_i, \qquad \gamma = \sum_{i=1}^{s} \bar{t}_i k_0$$

Now we apply Lemma 2.1, considering:

- the rows of matrix A are the vectors $[-\nabla g_j(x_0)]$, $j \in J(x_0)$;
- $c = \alpha$;
- $M_r^B = \begin{cases} \lambda^2 B_r & \text{if } r \in \overline{\mathcal{B}}(x_0) \\ 0 & \text{if } r \in \mathcal{B}(x_0) \end{cases}$ and $M_q^D = \begin{cases} \gamma^2 D_q & \text{if } q \in \overline{\mathcal{D}}(x_0) \\ 0 & \text{if } q \in \mathcal{D}(x_0) \end{cases}$.

It follows that there exist the scalars $\bar{\mu}_j \ge 0$, $j \in J(x_0)$, and the vectors $\bar{w}_r \in \mathbb{R}^n$, $r \in \overline{\mathcal{B}}(x_0)$, $\bar{v}_q \in \mathbb{R}^n$, $q \in \overline{\mathcal{D}}(x_0)$, such that

$$-\sum_{j \in J(x_0)} \bar{\mu}_j \nabla g_j(x_0) = c + \sum_{r \in \overline{\mathcal{B}}(x_0)} M_r^B \bar{w}_r + \sum_{q \in \overline{\mathcal{D}}(x_0)} M_q^D \bar{v}_q \tag{3.10}$$

and

$$\begin{aligned} \bar{w}_r^\top M_r^B \bar{w}_r \le 1, \qquad & r \in \overline{\mathcal{B}}(x_0) \\ \bar{v}_q^\top M_q^D \bar{v}_q \le 1, \qquad & q \in \overline{\mathcal{D}}(x_0) \end{aligned} \tag{3.11}$$

Since $g_j(x_0) = 0$ for $j \in J(x_0)$, we have: $\bar{\mu}_j g_j(x_0) = 0$ for $j \in J(x_0)$. If $j \notin J(x_0)$, we put $\bar{\mu}_j = 0$. It follows:

$$\sum_{j=1}^{p} \bar{\mu}_j g_j(x_0) = 0$$

which shows that relation (3.3) holds.

Now we define

$$w_r = \begin{cases} \dfrac{x_0}{\sqrt{x_0^\top B_r x_0}}, & \text{if } r \in \mathcal{B}(x_0) \\ \lambda \bar{w}_r, & \text{if } r \in \overline{\mathcal{B}}(x_0) \end{cases}$$

$$v_q = \begin{cases} \dfrac{x_0}{\sqrt{x_0^\top D_q x_0}}, & \text{if } q \in \mathcal{D}(x_0) \\ \gamma \bar{v}_q, & \text{if } q \in \overline{\mathcal{D}}(x_0) \end{cases}$$

With this notations, equality (3.10) yields relation (3.1).

From (3.11) we get: $w_r^\top B_r w_r \le 1$ for any $r = \overline{1,\beta}$. Further, if $r \in \overline{\mathcal{B}}(x_0)$, we have $x_0^\top B_r x_0 = 0$, which implies $B_r x_0 = 0$, and then $\sqrt{x_0^\top B_r x_0} = 0 = x_0^\top B_r w_r$. If $r \in \mathcal{B}(x_0)$, we obviously have $x_0^\top B_r w_r = \sqrt{x_0^\top B_r x_0}$. The same arguments apply to matrices D_q, so relation (3.5) holds. Therefore the theorem is proved. ■

For convenience, if a point $x_0 \in \mathcal{P}$ has the property that the vectors $\nabla g_j(x_0)$, $j \in J(x_0)$, are linear independent and the set $Z_{\bar{y}}(x_0) = \emptyset$, then we say that $x_0 \in \mathcal{P}$ satisfy a *constraint qualification.*

The results of Theorems 3.1 and 3.2 are the necessary conditions for the optimal solution of problem (P). Actually, the conditions (3.1) - (3.5) are also the sufficient optimality conditions for (P), for which we state the following result involving generalized invex functions, which are weaker assumptions than Lai et al. use in [14].

Theorem 3.3 (Sufficient Conditions) *Let $x_0 \in \mathcal{P}$ be a feasible solution of (P) and there exist a positive integer s, $1 \le s \le n+1$, $\bar{y}_i \in \underline{Y}(x_0)$, $i = \overline{1,s}$, $k_0 \in \mathbb{R}_+$, defined by (2.1), $\bar{t} \in \mathbb{R}_+^s$, $w_r \in \mathbb{R}^n$, $r = \overline{1,\beta}$, $v_q \in \mathbb{R}^n$, $q = \overline{1,\delta}$, and $\bar{\mu} \in \mathbb{R}_+^p$ such that the relations (3.1) - (3.5) are satisfied. If any one of the following four conditions holds:*

(a) $f(\cdot,\bar{y}_i) + \sum_{r=1}^{\beta} (\cdot)^\top B_r w_r$ is (η,ρ_i,θ)-invex, $h(\cdot,\bar{y}_i) - \sum_{q=1}^{\delta} (\cdot)^\top D_q v_q$ is (η,ρ_i',θ)-incave for $i = \overline{1,s}$, $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ is (η,ρ_0,θ)-invex, and $\rho_0 + \sum_{i=1}^{s} \bar{t}_i(\rho_i + \rho_i' k_0) \ge 0$,

(b) $\overline{\Phi}(\cdot) \overset{def}{=} \sum_{i=1}^{s} \bar{t}_i \left[f(\cdot, \bar{y}_i) + \sum_{r=1}^{\beta} (\cdot)^\top B_r w_r - k_0 \left(h(\cdot, \bar{y}_i) - \sum_{q=1}^{\delta} (\cdot)^\top D_q v_q \right) \right]$ is (η, ρ, θ)-invex and $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ is (η, ρ_0, θ)-invex, and $\rho + \rho_0 \geq 0$,

(c) $\overline{\Phi}(\cdot)$ is (η, ρ, θ)-pseudo-invex and $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ is (η, ρ_0, θ)-quasi-invex, and $\rho + \rho_0 \geq 0$,

(d) $\overline{\Phi}(\cdot)$ is (η, ρ, θ)-quasi-invex and $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ is strictly (η, ρ_0, θ)-pseudo-invex, and $\rho + \rho_0 \geq 0$,

then x_0 is an optimal solution of (P).

Proof. On contrary, let us suppose that x_0 is not an optimal solution of (P). Then there exists an $x_1 \in \mathcal{P}$ such that

$$\sup_{y \in Y} \frac{f(x_1, y) + \sum_{r=1}^{\beta} \sqrt{x_1^\top B_r x_1}}{h(x_1, y) - \sum_{q=1}^{\delta} \sqrt{x_1^\top D_q x_1}} < \sup_{y \in Y} \frac{f(x_0, y) + \sum_{r=1}^{\beta} \sqrt{x_0^\top B_r x_0}}{h(x_0, y) - \sum_{q=1}^{\delta} \sqrt{x_0^\top D_q x_0}}$$

We note that, for $\bar{y}_i \in Y(x_0)$, $i = \overline{1, s}$, we have

$$\sup_{y \in Y} \frac{f(x_0, y) + \sum_{r=1}^{\beta} \sqrt{x_0^\top B_r x_0}}{h(x_0, y) - \sum_{q=1}^{\delta} \sqrt{x_0^\top D_q x_0}} = \frac{f(x_0, \bar{y}_i) + \sum_{r=1}^{\beta} \sqrt{x_0^\top B_r x_0}}{h(x_0, \bar{y}_i) - \sum_{q=1}^{\delta} \sqrt{x_0^\top D_q x_0}} = k_0,$$

and

$$\frac{f(x_1, \bar{y}_i) + \sum_{r=1}^{\beta} \sqrt{x_1^\top B_r x_1}}{h(x_1, \bar{y}_i) - \sum_{q=1}^{\delta} \sqrt{x_1^\top D_q x_1}} \leq \sup_{y \in Y} \frac{f(x_1, y) + \sum_{r=1}^{\beta} \sqrt{x_1^\top B_r x_1}}{h(x_1, y) - \sum_{q=1}^{\delta} \sqrt{x_1^\top D_q x_1}}.$$

Thus, we have

$$\frac{f(x_1, \bar{y}_i) + \sum_{r=1}^{\beta} \sqrt{x_1^\top B_r x_1}}{h(x_1, \bar{y}_i) - \sum_{q=1}^{\delta} \sqrt{x_1^\top D_q x_1}} < k_0, \quad \text{for } i = \overline{1, s}.$$

It follows that

$$f\left(x_1,\bar{y}_i\right)+\sum_{r=1}^{\beta}\sqrt{x_1^{\top}B_rx_1}-k_0\left(h\left(x_1,\bar{y}_i\right)-\sum_{q=1}^{\delta}\sqrt{x_1^{\top}D_qx_1}\right)<0,\ \text{for } i=\overline{1,s}. \quad (3.12)$$

Using the relations (2.3), (3.5), (3.12), (3.2), and (3.4), we obtain

$$\begin{aligned}
\overline{\Phi}\left(x_1\right) &= \sum_{i=1}^{s}\bar{t}_i\left[f\left(x_1,\bar{y}_i\right)+\sum_{r=1}^{\beta}x_1^{\top}B_rw_r-k_0\left(h\left(x_1,\bar{y}_i\right)-\sum_{q=1}^{\delta}x_1^{\top}D_qv_q\right)\right] \\
&\leq \sum_{i=1}^{s}\bar{t}_i\left[f\left(x_1,\bar{y}_i\right)+\sum_{r=1}^{\beta}\sqrt{x_1^{\top}B_rx_1}-k_0\left(h\left(x_1,\bar{y}_i\right)-\sum_{q=1}^{\delta}\sqrt{x_1^{\top}D_qx_1}\right)\right] \\
&< 0 = \\
&= \sum_{i=1}^{s}\bar{t}_i\left[f\left(x_0,\bar{y}_i\right)+\sum_{r=1}^{\beta}\sqrt{x_0^{\top}B_rx_0}-k_0\left(h\left(x_0,\bar{y}_i\right)-\sum_{q=1}^{\delta}\sqrt{x_0^{\top}D_qx_0}\right)\right] \\
&= \sum_{i=1}^{s}\bar{t}_i\left[f\left(x_0,\bar{y}_i\right)+\sum_{r=1}^{\beta}x_0^{\top}B_rw_r-k_0\left(h\left(x_0,\bar{y}_i\right)-\sum_{q=1}^{\delta}x_0^{\top}D_qv_q\right)\right] \\
&= \overline{\Phi}\left(x_0\right).
\end{aligned}$$

It follows that

$$\overline{\Phi}\left(x_1\right)<\overline{\Phi}\left(x_0\right). \quad (3.13)$$

1. If hypothesis (a) holds, then for $i=\overline{1,s}$, we have

$$\begin{aligned}
&f\left(x_1,\bar{y}_i\right)+\sum_{r=1}^{\beta}x_1^{\top}B_rw_r-f\left(x_0,\bar{y}_i\right)-\sum_{r=1}^{\beta}x_0^{\top}B_rw_r\geq \\
&\geq \eta\left(x_1,x_0\right)^{\top}\left(\nabla f\left(x_0,\bar{y}_i\right)+\sum_{r=1}^{\beta}B_rw_r\right)+\rho_i\theta\left(x_1,x_0\right),
\end{aligned} \quad (3.14)$$

and

$$\begin{aligned}
&-h\left(x_1,\bar{y}_i\right)+\sum_{q=1}^{\delta}x_1^{\top}D_qv_q+h\left(x_0,\bar{y}_i\right)-\sum_{q=1}^{\delta}x_0^{\top}D_qv_q\geq \\
&\geq \eta\left(x_1,x_0\right)^{\top}\left(-\nabla h\left(x_0,\bar{y}_i\right)+\sum_{q=1}^{\delta}D_qv_q\right)+\rho_i'\theta\left(x_1,x_0\right).
\end{aligned} \quad (3.15)$$

Now, multiplying (3.14) by $\bar{t}_i$, (3.15) by $\bar{t}_ik_0$, and then sum up these inequalities, we obtain

$$\begin{aligned}&\overline{\Phi}(x_1)-\overline{\Phi}(x_0)\geq\\&\geq\eta(x_1,x_0)^{\top}\sum_{i=1}^{s}\bar{t}_i\left[\nabla f(x_0,\bar{y}_i)+\sum_{r=1}^{\beta}B_rw_r-k_0\left(\nabla h(x_0,\bar{y}_i)-\sum_{q=1}^{\delta}D_qv_q\right)\right]\\&\quad+\sum_{i=1}^{s}\bar{t}_i(\rho_i+k_0\rho_i')\theta(x_1,x_0).\end{aligned}$$

Further, by (3.1) and (η,ρ_0,θ)-invexity of $\sum_{j=1}^{p}\bar{\mu}_jg_j(\cdot)$, we get

$$\begin{aligned}\overline{\Phi}(x_1)-\overline{\Phi}(x_0)&\geq-\eta(x_1,x_0)^{\top}\sum_{j=1}^{p}\bar{\mu}_j\nabla g_j(x_0)+\sum_{i=1}^{s}\bar{t}_i(\rho_i+k_0\rho_i')\theta(x_1,x_0)\\&\geq-\sum_{j=1}^{p}\bar{\mu}_jg_j(x_1)+\sum_{j=1}^{p}\bar{\mu}_jg_j(x_0)+\\&\qquad+\left(\rho_0+\sum_{i=1}^{s}\bar{t}_i(\rho_i+k_0\rho_i')\right)\theta(x_1,x_0).\end{aligned}$$

Since $x_1\in\mathcal{P}$, we have $g_i(x_1)\leq 0$, $i=\overline{1,s}$, and using (3.3) it follows

$$\overline{\Phi}(x_1)-\overline{\Phi}(x_0)\geq\left(\rho_0+\sum_{i=1}^{s}\bar{t}_i(\rho_i+k_0\rho_i')\right)\theta(x_1,x_0)\geq 0,$$

which contradicts the inequality (3.13).

2. If the hypothesis (b) holds, we have

$$\begin{aligned}&\overline{\Phi}(x_1)-\overline{\Phi}(x_0)\geq\\&\geq\eta(x_1,x_0)^{\top}\sum_{i=1}^{s}\bar{t}_i\left[\nabla f(x_0,\bar{y}_i)+\sum_{r=1}^{\beta}B_rw_r-k_0\left(\nabla h(x_0,\bar{y}_i)-\sum_{q=1}^{\delta}D_qv_q\right)\right]\\&\quad+\rho\theta(x_1,x_0).\end{aligned}$$

Using relation (3.1) and the (η,ρ_0,θ)-invexity of $\sum_{j=1}^{p}\bar{\mu}_jg_j(\cdot)$, we obtain

$$\begin{aligned}\overline{\Phi}(x_1)-\overline{\Phi}(x_0)&\geq-\eta(x_1,x_0)^{\top}\sum_{j=1}^{p}\bar{\mu}_j\nabla g_j(x_0)+\rho\theta(x_1,x_0)\geq\\&\geq-\sum_{j=1}^{p}\bar{\mu}_jg_j(x_1)+\sum_{j=1}^{p}\bar{\mu}_jg_j(x_0)+(\rho+\rho_0)\theta(x_1,x_0)\geq\\&\geq(\rho+\rho_0)\theta(x_1,x_0)\geq 0,\end{aligned}$$

which contradicts the inequality (3.13).

3. If the hypothesis (c) holds, using the (η,ρ,θ)-pseudo-invexity of $\overline{\Phi}$, it follows from (3.13) that

$$\overline{\Phi}(x_1) < \overline{\Phi}(x_0) \implies \eta(x_1,x_0)^\top \nabla\overline{\Phi}(x_0) < -\rho\theta(x_1,x_0). \tag{3.16}$$

Using again relation (3.1), from (3.16) and $\rho+\rho_0 \geq 0$, we get

$$\eta(x_1,x_0)^\top \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(x_0) > \rho\theta(x_1,x_0) \geq -\rho_0\theta(x_1,x_0). \tag{3.17}$$

Since $x_1 \in \mathcal{P}$ imply $g_i(x_1) \leq 0$, $i=\overline{1,s}$, and $\bar{\mu} \in \mathbb{R}^p_+$, using (3.3) we have

$$\sum_{j=1}^{p} \bar{\mu}_j g_j(x_1) \leq 0 = \sum_{j=1}^{p} \bar{\mu}_j g_j(x_0). \tag{3.18}$$

Using the (η,ρ_0,θ)-quasi-invexity of $\sum_{j=1}^{p} \bar{\mu}_j g_j(\,\cdot\,)$, we get from the last relation

$$\eta(x_1,x_0)^\top \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(x_0) \leq \rho_0\theta(x_1,x_0)$$

which contradicts the inequality (3.17).

4. If the hypothesis (d) holds, the (η,ρ,θ)-quasi-invexity of $\overline{\Phi}$ imply

$$\overline{\Phi}(x_1) \leq \overline{\Phi}(x_0) \implies \eta(x_1,x_0)^\top \nabla\overline{\Phi}(x_0) \leq -\rho\theta(x_1,x_0).$$

From here, together with (3.1) and $\rho+\rho_0 \geq 0$, we have

$$\eta(x_1,x_0)^\top \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(x_0) \geq \rho\theta(x_1,x_0) \geq -\rho_0\theta(x_1,x_0). \tag{3.19}$$

Since (3.18) is true, the strictly (η,ρ,θ)-pseudo-invexity of $\sum_{j=1}^{p} \bar{\mu}_j g_j(\,\cdot\,)$ imply

$$\eta(x_1,x_0)^\top \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(x_0) < \rho_0\theta(x_1,x_0)$$

which contradicts the inequality (3.19).

Therefore the proof of the theorem is complete. ■

4. Duality

Let us consider the set $H(s,t,y)$ consisting of all $(z,\mu,k,v,w) \in \mathbb{R}^n \times \mathbb{R}^p_+ \times \mathbb{R}_+ \times \mathbb{R}^{n\delta} \times \mathbb{R}^{n\beta}$, where $v = (v_1, \cdots, v_\delta)$, $v_q \in \mathbb{R}^n$, $q = \overline{1,\delta}$, and $w = (w_1, \cdots, w_\beta)$, $w_r \in \mathbb{R}^n$, $r = \overline{1,\beta}$, which satisfy the following conditions:

$$\begin{gathered}\sum_{i=1}^{s} t_i \left[\nabla f(z,y_i) + \sum_{r=1}^{\beta} B_r w_r - k\left(\nabla h(z,y_i) - \sum_{q=1}^{\delta} D_q v_q\right)\right] + \\ + \sum_{j=1}^{p} \mu_j \nabla g_j(z) = 0,\end{gathered} \tag{4.1}$$

$$\sum_{i=1}^{s} t_i \left[f(z,y_i) + \sum_{r=1}^{\beta} z^\top B_r w_r - k\left(h(z,y_i) - \sum_{q=1}^{\delta} z^\top D_q v_q\right)\right] \geq 0, \tag{4.2}$$

$$\sum_{j=1}^{p} \mu_j g_j(z) \geq 0, \tag{4.3}$$

$$(s,t,y) \in K(z) \tag{4.4}$$

$$\begin{aligned} w_r^\top B_r w_r \leq 1, \quad r = \overline{1,\beta}, \\ v_q^\top D_q v_q \leq 1, \quad q = \overline{1,\delta}. \end{aligned} \tag{4.5}$$

The optimality conditions, stated in the preceding section for the minimax problem (P), suggest us to define the following dual problem:

$$\max_{(s,t,y)\in K(z)} \sup \{k \mid (z,u,k,v,w) \in H(s,t,y)\} \tag{DP}$$

If, for a triplet $(s,t,y) \in K(z)$, the set $H(s,t,y) = \emptyset$, then we define the supremum over $H(s,t,y)$ to be $-\infty$. Further, we denote

$$\Phi(\cdot) = \sum_{i=1}^{s} t_i \left[f(\cdot,y_i) + \sum_{r=1}^{\beta} (\cdot)^\top B_r w_r - k\left(h(\cdot,y_i) - \sum_{q=1}^{\delta} (\cdot)^\top D_q v_q\right)\right]$$

Now, we can state the following weak duality theorem for (P) and (DP).

Theorem 4.1 (Weak Duality) *Let $x \in \mathcal{P}$ be a feasible solution of (P) and (x,μ,k,v,w,s,t,y) be a feasible solution of (DP). If any of the following four conditions holds:*

(a) $f(\cdot,y_i) + \sum_{r=1}^{\beta} (\cdot)^\top B_r w_r$ is (η,ρ_i,θ)-invex, $h(\cdot,y_i) - \sum_{q=1}^{\delta} (\cdot)^\top D_q v_q$ is (η,ρ_i',θ)-incave for $i = \overline{1,s}$, $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ is (η,ρ_0,θ)-invex, and $\rho_0 + \sum_{i=1}^{s} t_i(\rho_i + \rho_i' k) \geq 0$,

(b) $\Phi(\cdot)$ is (η,ρ,θ)-invex and $\sum\limits_{j=1}^{p}\mu_j g_j(\cdot)$ is (η,ρ_0,θ)-invex, and $\rho+\rho_0\geq 0$,

(c) $\Phi(\cdot)$ is (η,ρ,θ)-pseudo-invex and $\sum\limits_{j=1}^{p}\mu_j g_j(\cdot)$ is (η,ρ_0,θ)-quasi-invex, and $\rho+\rho_0\geq 0$,

(d) $\Phi(\cdot)$ is (η,ρ,θ)-quasi-invex and $\sum\limits_{j=1}^{p}\mu_j g_j(\cdot)$ is strictly (η,ρ_0,θ)-pseudo-invex, and $\rho+\rho_0\geq 0$,

then

$$\sup_{y\in Y}\frac{f(x,y)+\sum\limits_{r=1}^{\beta}\sqrt{x^\top B_r x}}{h(x,y)-\sum\limits_{q=1}^{\delta}\sqrt{x^\top D_q x}}\geq k \tag{4.6}$$

Proof. If we suppose, on contrary, that

$$\sup_{y\in Y}\frac{f(x,y)+\sum\limits_{r=1}^{\beta}\sqrt{x^\top B_r x}}{h(x,y)-\sum\limits_{q=1}^{\delta}\sqrt{x^\top D_q x}}< k$$

then we have, for all $y\in Y$,

$$f(x,y)+\sum_{r=1}^{\beta}\sqrt{x^\top B_r x}-k\left(h(x,y)-\sum_{q=1}^{\delta}\sqrt{x^\top D_q x}\right)<0.$$

It follows that, for $t_i\geq 0$, $i=\overline{1,s}$, with $\sum\limits_{i=1}^{s}t_i=1$,

$$t_i\left[f(x,y)+\sqrt{x^\top B_r x}-k\left(h(x,y)-\sqrt{x^\top D_q x}\right)\right]\leq 0,\quad i=\overline{1,s}, \tag{4.7}$$

with at least one strict inequality, because $t=(t_1,\cdots,t_s)\neq 0$.

Taking into account the relations (2.3), (4.5), (4.7) and (4.2), we have

$$\begin{aligned}\Phi(x) &= \sum_{i=1}^{s}t_i\left[f(x,y_i)+\sum_{r=1}^{\beta}x^\top B_r w_r-k\left(h(x,y_i)-\sum_{q=1}^{\delta}x^\top D_q v_q\right)\right]\\ &\leq \sum_{i=1}^{s}t_i\left[f(x,y_i)+\sum_{r=1}^{\beta}\sqrt{x^\top B_r x}-k\left(h(x,y_i)-\sum_{q=1}^{\delta}\sqrt{x^\top D_q x}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&< \quad 0 \leq \sum_{i=1}^{s} t_i \left[f(z, y_i) + \sum_{r=1}^{\beta} z^\top B_r w_r - k \left(h(z, y_i) - \sum_{q=1}^{\delta} z^\top D_q v_q \right) \right] \\
&= \quad \Phi(z),
\end{aligned}
$$

that is

$$
\Phi(x) < \Phi(z). \tag{4.8}
$$

1. If hypothesis (a) holds, then for $i = \overline{1, s}$, we have

$$
\begin{gathered}
f(x, y_i) + \sum_{r=1}^{\beta} x^\top B_r w_r - f(z, y_i) - \sum_{r=1}^{\beta} z^\top B_r w_r \geq \\
\geq \eta(x, z)^\top \left(\nabla f(z, y_i) + \sum_{r=1}^{\beta} B_r w_r \right) + \rho_i \theta(x, z),
\end{gathered} \tag{4.9}
$$

and

$$
\begin{gathered}
-h(x, y_i) + \sum_{q=1}^{\delta} x^\top D_q v_q + h(z, y_i) - \sum_{q=1}^{\delta} z^\top D_q v_q \geq \\
\geq \eta(x, z)^\top \left(-\nabla h(z, y_i) + \sum_{q=1}^{\delta} D_q v_q \right) + \rho_i' \theta(x, z).
\end{gathered} \tag{4.10}
$$

Now, multiplying (4.9) by t_i, (4.10) by $t_i k$, and then sum up these inequalities, we obtain

$$
\begin{aligned}
&\Phi(x) - \Phi(z) \geq \\
&\geq \eta(x, z)^\top \sum_{i=1}^{s} t_i \left[\nabla f(z, y_i) + \sum_{r=1}^{\beta} B_r w_r - k \left(\nabla h(z, y_i) - \sum_{q=1}^{\delta} D_q v_q \right) \right] \\
&\quad + \sum_{i=1}^{s} t_i \left(\rho_i + k \rho_i' \right) \theta(x, z).
\end{aligned}
$$

Further, by (4.1) and (η, ρ_0, θ)-invexity of $\sum_{j=1}^{p} \mu_j g_j(\cdot)$, we get

$$
\begin{aligned}
&\Phi(x) - \Phi(z) \geq -\eta(x, z)^\top \sum_{j=1}^{p} \mu_j \nabla g_j(z) + \sum_{i=1}^{s} t_i \left(\rho_i + k \rho_i' \right) \theta(x, z) \\
&\geq -\sum_{j=1}^{p} \mu_j g_j(x) + \sum_{j=1}^{p} \mu_j g_j(z) + \left(\rho_0 + \sum_{i=1}^{s} t_i \left(\rho_i + k \rho_i' \right) \right) \theta(x, z).
\end{aligned}
$$

Since $x \in \mathcal{P}$, we have $g_i(x) \leq 0$, $i = \overline{1, s}$, and using (4.3) it follows

$$
\Phi(x) - \Phi(z) \geq \left(\rho_0 + \sum_{i=1}^{s} t_i \left(\rho_i + k \rho_i' \right) \right) \theta(x, z) \geq 0,
$$

which contradicts the inequality (4.8). Hence, the inequality (4.6) is true.

2. Similarly, one can prove the case of hypothesis (b).

3. If hypothesis (c) holds, using the (η, ρ, θ)-pseudo-invexity of Φ, we get from (4.8) that

$$\eta(x,z)^\top \nabla\Phi(z) < -\rho\theta(x,z) \tag{4.11}$$

Consequently, relations (4.1), (4.11) and $\rho+\rho_0 \geq 0$, yield

$$\eta(x,z)^\top \sum_{j=1}^{p} \mu_j \nabla g_j(z) > \rho\theta(x,z) \geq -\rho_0\theta(x,z). \tag{4.12}$$

Because $x \in \mathcal{P}$, $\mu \in \mathbb{R}_+^p$, and (4.3), we have

$$\sum_{j=1}^{p} \mu_j g_j(x) \leq 0 = \sum_{j=1}^{p} \mu_j g_j(z).$$

Using the (η, ρ_0, θ)-quasi-invexity of $\sum_{j=1}^{p} \mu_j g_j(\cdot)$, we get from the last relation

$$\eta(x,z)^\top \sum_{j=1}^{p} \mu_j \nabla g_j(z) \leq \rho_0\theta(x,z),$$

which contradicts the inequality (4.12).

4. The result under the hypothesis (d) follows similarly like in step 3.

Therefore the proof of the theorem is complete. ■

Theorem 4.2 (Strong Duality) *Let x^* be an optimal solution of problem (P). Assume that x^* satisfies a constraint qualification for problem (P). Then there exist $(s^*, t^*, y^*) \in K(x^*)$ and $(x^*, \mu^*, k^*, v^*, w^*) \in H(s^*, t^*, y^*)$ such that $(x^*, \mu^*, k^*, v^*, w^*, s^*, t^*, y^*)$ is a feasible solution of (DP). If the hypotheses of Theorem 4.1 are also satisfied, then $(x^*, \mu^*, k^*, v^*, w^*, s^*, t^*, y^*)$ is an optimal solution for (DP), and both problems (P) and (DP) have the same optimal values.*

Proof. By Theorems 3.1 and 3.2, there exist $(s^*, t^*, y^*) \in K(x^*)$ and $(x^*, \mu^*, k^*, v^*, w^*) \in H(s^*, t^*, y^*)$ such that $(x^*, \mu^*, k^*, v^*, w^*, s^*, t^*, y^*)$ is a feasible solution of (DP), and

$$k^* = \frac{f(x^*, y_i^*) + \sum_{r=1}^{\beta} \sqrt{(x^*)^\top B_r x^*}}{h(x^*, y_i^*) - \sum_{q=1}^{\delta} \sqrt{(x^*)^\top D_q x^*}}.$$

The optimality of this feasible solution for (DP) follows from Theorem 4.1. ■

Theorem 4.3 (Strict Converse Duality) *Let x^* and $(\bar{z}, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y})$ be the optimal solutions of (P) and (DP), respectively, and that the hypotheses of Theorem 4.2 are fulfilled. If any one of the following three conditions holds:*

(a) one of $f\left(\,\cdot\,,\bar{y}_i\right)+\sum\limits_{r=1}^{\beta}\left(\,\cdot\,\right)^{\top}B_r\bar{w}_r$ is strictly (η,ρ_i,θ)-invex, $h\left(\,\cdot\,,\bar{y}_i\right)-\sum\limits_{q=1}^{\delta}\left(\,\cdot\,\right)^{\top}D_q\bar{v}_q$ is strictly (η,ρ_i',θ)-incave for $i=\overline{1,s}$, or $\sum\limits_{j=1}^{p}\bar{\mu}_j g_j\left(\,\cdot\,\right)$ is strictly (η,ρ_0,θ)-invex, and $\rho_0+\sum\limits_{i=1}^{s}\bar{t}_i\left(\rho_i+\rho_i'\bar{k}\right)\geq 0$;

(b) either $\sum\limits_{i=1}^{s}\bar{t}_i\left[f\left(\,\cdot\,,\bar{y}_i\right)+\sum\limits_{r=1}^{\beta}\left(\,\cdot\,\right)^{\top}B_r\bar{w}_r-\bar{k}\left(h\left(\,\cdot\,,\bar{y}_i\right)-\sum\limits_{q=1}^{\delta}\left(\,\cdot\,\right)^{\top}D_q\bar{v}_q\right)\right]$ is strictly (η,ρ,θ)-invex or $\sum\limits_{j=1}^{p}\bar{\mu}_j g_j\left(\,\cdot\,\right)$ is strictly (η,ρ_0,θ)-invex, and $\rho+\rho_0\geq 0$;

(c) $\sum\limits_{i=1}^{s}\bar{t}_i\left[f\left(\,\cdot\,,\bar{y}_i\right)+\sum\limits_{r=1}^{\beta}\left(\,\cdot\,\right)^{\top}B_r\bar{w}_r-\bar{k}\left(h\left(\,\cdot\,,\bar{y}_i\right)-\sum\limits_{q=1}^{\delta}\left(\,\cdot\,\right)^{\top}D_q\bar{v}_q\right)\right]$ is strictly (η,ρ,θ)-pseudo-invex and $\sum\limits_{j=1}^{p}\bar{\mu}_j g_j\left(\,\cdot\,\right)$ is (η,ρ_0,θ)-quasi-invex, and $\rho+\rho_0\geq 0$;

then $x^*=\bar{z}$, that is, $\bar{z}$ is an optimal solution for problem (P) and

$$\sup_{y\in Y}\frac{f\left(\bar{z},y\right)+\sum\limits_{r=1}^{\beta}\sqrt{\bar{z}^{\top}B_r\bar{z}}}{h\left(\bar{z},y\right)-\sum\limits_{q=1}^{\delta}\sqrt{\bar{z}^{\top}D_q\bar{z}}}=\bar{k}.$$

Proof. Suppose on the contrary that $x^*\neq\bar{z}$. From Theorem 4.2 we know that there exist $(s^*,t^*,y^*)\in K\left(x^*\right)$ and $(x^*,\mu^*,k^*,v^*,w^*)\in H\left(s^*,t^*,y^*\right)$ such that $(x^*,\mu^*,k^*,v^*,w^*,s^*,t^*,y^*)$ is a feasible solution for (DP) with the optimal value

$$k^*=\sup_{y\in Y}\frac{f\left(x^*,y\right)+\sum\limits_{r=1}^{\beta}\sqrt{\left(x^*\right)^{\top}B_r x^*}}{h\left(x^*,y\right)-\sum\limits_{q=1}^{\delta}\sqrt{\left(x^*\right)^{\top}D_q x^*}}.$$

Now, we proceed similarly as in the proof of Theorem 4.1, replacing x by x^* and (z,μ,k,v,w,s,t,y) by $\left(\bar{z},\bar{\mu},\bar{k},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y}\right)$, so that we arrive at the strict inequality

$$\sup_{y\in Y}\frac{f\left(x^*,y\right)+\sum\limits_{r=1}^{\beta}\sqrt{\left(x^*\right)^{\top}B_r x^*}}{h\left(x^*,y\right)-\sum\limits_{q=1}^{\delta}\sqrt{\left(x^*\right)^{\top}D_q x^*}}>\bar{k}.$$

But this contradicts the fact

$$\sup_{y\in Y}\frac{f\left(x^{*},y\right)+\sqrt{\left(x^{*}\right)^{\top}B_{r}x^{*}}}{h\left(x^{*},y\right)-\sqrt{\left(x^{*}\right)^{\top}D_{q}x^{*}}}=k^{*}=\bar{k},$$

and we conclude that $x^{*}=\bar{z}$. Hence, the proof of the theorem is complete. ■

5. Special Cases

If we consider special cases of the results presented in this paper, we may retrieve some previous results obtained by other authors.

1. If we consider $\beta=\delta=1$, we obtain the results obtained by Lai et al. [14].
2. If $B_r=0$, $r=\overline{1,\beta}$, and $D_q=0$, $q=\overline{1,\delta}$, we obtain the results of Chandra and Kumar [7], respectively that of Liu and Wu [16].
3. If the set Y is a singleton, $\beta=1$, $h\equiv 1$ and $D_q=0$, $q=\overline{1,\delta}$, we obtain the results presented respectively in Mond [18], Chandra et al. [5], Preda [20], Zhang and Mond [30], Preda and Köller [22].
4. If the set Y is a singleton and $\beta=\delta=0$ (that is, we have no square root terms), then problem (P) becomes the standard fractional programming problem and the dual problem (DP) reduce to the well known dual of Schaible [24].
5. For the case of the generalized fractional programming [2, 3, 6, 10, 12], the set Y can be taken as the simplex $Y=\left\{y\in\mathbb{R}^{m}\;\middle|\;y_i\geq 0,\sum\limits_{i=1}^{m}y_i=1\right\}$, $B_r=0$, $r=\overline{1,\beta}$, and $D_q=0$, $q=\overline{1,\delta}$, and

$$\frac{f\left(x,y\right)}{h\left(x,y\right)}=\frac{\sum\limits_{i=1}^{m}y_if_i\left(x\right)}{\sum\limits_{i=1}^{m}y_ih_i\left(x\right)}.$$

In this case the dual (DP) reduces to the dual problem of [2].

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